The Question Bank of

Pure Mathematics 3

for CAIE 9709 paper 3.

v1.0

Edited by Thoridal

Instructions for Use

- This question bank is organized by chapter for systematic revision.
- This question bank is complied based on the 26-27 CAIE Pure Mathematics 3 syllabus, which is included as appendix.
- Each question includes its source for reference.
- Mark schemes are provided in the separate answer booklet.
- The formula sheet (MF19) is included as appendix.
- Use this resource for targeted practice and exam preparation.

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Chapter 1

Algebra (modulus functions)

1. [9709/s25/31/q1]

(a) Sketch the graph of $y = |2x - 3|$. [1]

(b) Solve the inequality $3x - 1 < |2x - 3|$. [2]

2. [9709/s25/33/q1]

(a) Sketch the graph of $y = |3x - 2a|$, where a is a positive constant. [1]

(b) Hence or otherwise solve the inequality $|3x - 2a| < x + 5a$. [3]

3. [9709/w25/32/q1]

(a) Sketch the graph of $y = |x + 3a|$, where a is a positive constant. [1]

(b) Hence or otherwise solve the inequality $|x + 3a| > a - 2x$. [2]

4. [9709/w25/33/q1]

Solve the inequality $|3x+2| < 3|2x-1|$. [4]

5. [9709/w25/35/q1]

(a) Sketch the graph of $y = |3x - 6|$. [1]

(b) Solve the inequality $5x - 3 < |3x - 6|$. [3]

6. [9709/s24/32/q1]

(a) Sketch the graph of $y = |x - 2a|$, where a is a positive constant. [1]

(b) Solve the inequality $2x - 3a < |x - 2a|$. [2]

7. [9709/s23/31/q2]

(a) Sketch the graph of $y = |2x + 3|$. [1]

(b) Solve the inequality $3x + 8 > |2x + 3|$. [3]

8. [9709/s23/32/q1]

Solve the inequality $|5x - 3| < 2|3x - 7|$. [4]

9. [9709/w23/32/q1]

(a) Sketch the graph of $y = |4x - 2|$. [1]

(b) Solve the inequality $1 + 3x < |4x - 2|$. [4]

10. [9709/m22/32/q1]

Solve the inequality $|2x + 3| > 3|x + 2|$. [4]

11. [9709/s22/33/q1]

Find, in terms of a, the set of values of x satisfying the inequality

$$2|3x + a| < |2x + 3a|,$$

where a is a positive constant. [4]

12. [9709/w22/31/q1]

(a) Sketch the graph of $y = |2x + 1|$. [1]

(b) Solve the inequality $3x + 5 < |2x + 1|$. [3]

13. [9709/s21/31/q1]

Solve the inequality $2|3x - 1| < |x + 1|$. [4]

14. [9709/s21/32/q1]

Solve the inequality $|2x - 1| < 3|x + 1|$. [4]

15. [9709/w21/32/q2]

Solve the inequality $|3x - a| > 2|x + 2a|$, where a is a positive constant. [4]

16. [9709/w21/33/q2]

(a) Sketch the graph of $y = |2x - 3|$. [1]

(b) Solve the inequality $|2x - 3| < 3x + 2$. [3]

17. [9709/m20/32/q1]

(a) Sketch the graph of $y = |x - 2|$. [1]

(b) Solve the inequality $|x - 2| < 3x - 4$. [3]

18. [9709/s20/33/q1]

Solve the inequality $|2x - 1| > 3|x + 2|$. [4]

19. [9709/w20/31/q1]

Solve the inequality $2 - 5x > 2|x - 3|$. [4]

20. [9709/w19/31/q2]

Solve the inequality $|2x - 3| > 4|x + 1|$. [4]

21. [9709/w19/33/q1]

Solve the inequality $2|x+2| > |3x-1|$. [4]

22. [9709/w18/31/q1]

Find the set of values of x satisfying the inequality $2|2x - a| < |x + 3a|$, where a is a positive constant. [4]

23. [9709/w18/32/q1]

Solve the inequality $3|2x - 1| > |x + 4|$. [4]

24. [9709/m17/32/q2]

Solve the inequality $|x - 4| < 2|3x + 1|$. [4]

25. [9709/s17/31/q1]

Solve the inequality $|2x + 1| < 3|x - 2|$. [4]

26. [9709/s17/32/q2]

Solve the inequality $|x - 3| < 3x - 4$. [4]

27. [9709/s16/33/q1]

Solve the inequality $2|x-2| > |3x+1|$. [4]

28. [9709/s15/33/q2]

Solve the inequality $|x-2| > 2x-3$. [4]

29. [9709/w15/31/q1]

Solve the inequality $|2x - 5| > 3|2x + 1|$. [4]

Chapter 2

Algebra (factor theorem and remainder theorem)

1. [9709/m25/32/q9]

The polynomial $6x^3 + ax^2 + bx + 9$ is denoted by $\mathrm{p}(x)$, where a and b are constants. It is given that $(x-3)$ is a factor of $\mathrm{p}(x)$, and when the first derivative $\mathrm{p}'(x)$ is divided by $(x-3)$ the remainder is 72.

(a) Find the values of a and b. [5]

(b) When a and b have the values found in part **(a)**, factorise $\mathrm{p}(x)$ completely. [3]

(c) Hence solve the inequality $\mathrm{p}(x) < 0$. [2]

2. [9709/s25/31/q5]

The polynomial $3x^3 + pax^2 + 7a^2x + qa^3$ is denoted by $f(x)$, where p, q and a are constants and $a \neq 0$.

When $f(x)$ is divided by $(x+2a)$ the remainder is $-22a^3$. When $f(x)$ is divided by $(3x-a)$ the remainder is $-a^3$.

Find the values of p and q. [5]

3. [9709/s25/31/q10]

(a) Find the quotient and remainder when $x^3+5x^2-2x-15$ is divided by x^2-3. [3]

(b) The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^3+5x^2-2x-15}{6y(x^2-3)}.$$

It is given that $y=2$ when $x=2$.

Solve the differential equation to obtain an expression for y^2 in terms of x. [5]

4. [9709/s25/32/q10]

(a) Find the quotient and remainder when x^2 is divided by $1+4x^2$. [2]

(b) Find the exact value of $\int_0^{0.5} x\tan^{-1}(2x)\,dx$. [6]

5. [9709/w25/32/q5]

(a) It is given that $f(x) = (x-a)^2 g(x)$, where $f(x)$ and $g(x)$ are polynomials.

Show that $(x-a)$ is a factor of $f'(x)$. [2]

(b) It is given that $(x-3)^2$ is a factor of $2x^3 - 4x^2 + px + q$, where p and q are constants.

Find the values of p and q. [5]

6. [9709/w25/33/q2]

Find the quotient and the remainder when $3x^4 - 2x^2$ is divided by $x + 1$. [3]

7. [9709/m24/32/q1]

Find the quotient and remainder when $x^4 - 3x^3 + 9x^2 - 12x + 27$ is divided by $x^2 + 5$. [3]

8. [9709/s24/33/q7]

Let $f(x) = 8x^3 + 54x^2 - 17x - 21$.

(a) Show that $x+7$ is a factor of $f(x)$. [1]

(b) Find the quotient when $f(x)$ is divided by $x+7$. [2]

(c) Hence solve the equation

$$8\cos^3\theta + 54\cos^2\theta - 17\cos\theta - 21 = 0,$$

for $0° \leqslant \theta \leqslant 360°$. [3]

9. [9709/w24/31/q1]

The polynomial $4x^3 + ax^2 + 5x + b$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(2x+1)$ is a factor of $\mathrm{p}(x)$. When $\mathrm{p}(x)$ is divided by $(x-4)$ the remainder is equal to 3 times the remainder when $\mathrm{p}(x)$ is divided by $(x-2)$.

Find the values of a and b. [5]

10. [9709/w24/33/q9]

(a) Find the quotient and remainder when $x^4 + 16$ is divided by $x^2 + 4$. [3]

(b) Hence show that $\int_2^{2\sqrt{3}} \frac{x^4 + 16}{x^2 + 4} \, dx = \frac{4}{3}(\pi + 4)$. [5]

11. [9709/m23/32/q3]

The polynomial $2x^4 + ax^3 + bx - 1$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $x^2 - x + 1$ the remainder is $3x + 2$.

Find the values of a and b. [5]

12. [9709/s23/31/q10]

The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $p(x)$.

(a) Show that $(x + 3)$ is a factor of $p(x)$. [2]

(b) Show that $z = -1 + 2\sqrt{6}i$ is a root of $p(z) = 0$. [3]

(c) Hence find the complex numbers z which are roots of $p(z^2) = 0$. [7]

13. [9709/s23/33/q2]

Find the quotient and remainder when $2x^4 - 27$ is divided by $x^2 + x + 3$. [3]

14. [9709/w23/32/q3]

The polynomial $2x^3 + ax^2 - 11x + b$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 12.

Find the values of a and b. [5]

15. [9709/w23/33/q3]

The polynomial $2x^3 + ax^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $(x + 2)$ the remainder is -38 and when $p(x)$ is divided by $(2x - 1)$ the remainder is $\frac{19}{2}$.

Find the values of a and b. [5]

16. [9709/m22/32/q8.a]

(a) Find the quotient and remainder when $8x^3 + 4x^2 + 2x + 7$ is divided by $4x^2 + 1$. [3]

(b) Hence find the exact value of $\int_0^{\frac{1}{2}} \frac{8x^3 + 4x^2 + 2x + 7}{4x^2 + 1}\,dx$. [5]

17. [9709/s22/31/q5]

The polynomial $ax^3 - 10x^2 + bx + 8$, where a and b are constants, is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of both $p(x)$ and $p'(x)$.

(a) Find the values of a and b. [5]

(b) When a and b have these values, factorise $p(x)$ completely. [3]

18. [9709/s22/32/q3]

The polynomial $ax^3 + x^2 + bx + 3$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 5.

Find the values of a and b. [5]

19. [9709/w22/32/q2]

The polynomial $2x^3 - x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $(2x + 3)$ is a factor of $p(x)$.

(a) Find the value of a. [2]

(b) When a has this value, solve the inequality $p(x) < 0$. [4]

20. [9709/m21/32/q2]

The polynomial $ax^3 + 5x^2 - 4x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 2.

Find the values of a and b. [5]

21. [9709/w21/33/q1]

Find the quotient and remainder when $2x^4 + 1$ is divided by $x^2 - x + 2$. [3]

22. [9709/s20/31/q5.a]

(a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

(b) Using your answer to part **(a)**, find the exact value of $\displaystyle\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\, dx$. [5]

23. [9709/s20/32/q1]

Find the quotient and remainder when $6x^4 + x^3 - x^2 + 5x - 6$ is divided by $2x^2 - x + 1$. [3]

24. [9709/w19/32/q3]

The polynomial $x^4 + 3x^3 + ax + b$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $x^2 + x - 1$ the remainder is $2x + 3$. Find the values of a and b. [5]

25. [9709/s18/31/q4]

The polynomial $x^4 + 2x^3 + ax + b$, where a and b are constants, is divisible by $x^2 - x + 1$. Find the values of a and b. [5]

26. [9709/w17/31/q1]

Find the quotient and remainder when x^4 is divided by $x^2 + 2x - 1$. [3]

27. [9709/m16/32/q4]

The polynomial $4x^3 + ax + 2$, where a is a constant, is denoted by $\mathrm{p}(x)$. It is given that $(2x + 1)$ is a factor of $\mathrm{p}(x)$.

(i) Find the value of a. [2]

(ii) When a has this value,

(a) factorise $\mathrm{p}(x)$, [2]

(b) solve the inequality $\mathrm{p}(x) > 0$, justifying your answer. [3]

28. [9709/w16/33/q4]

The polynomial $4x^4 + ax^2 + 11x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $x^2 - x + 2$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the real roots of the equation $p(x) = 0$. [2]

29. [9709/w15/31/q6]

The polynomial $8x^3 + ax^2 + bx - 1$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(2x + 1)$ the remainder is 1.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $p(x)$ completely. [3]

Chapter 3

Algebra (partial fractions and binomial expansions)

1. [9709/m25/32/q10]

Let $\mathrm{f}(x) = \dfrac{-7x^2 + 2x - 6}{(1+x)(4+x^2)}$.

(a) Express $\mathrm{f}(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\displaystyle\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$. Give your answer in the form $a\pi - \ln b$, where a and b are constants. [6]

2. [9709/s25/32/q2]

(a) Expand $(6-x)(1-2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

(b) State the set of values of x for which the expansion is valid. [1]

3. [9709/s25/33/q7]

Let $f(x) = \dfrac{3a-5x}{(3a+2x)(2a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [4]

(c) State the set of values of x for which the expansion in part **(b)** is valid. [1]

4. [9709/s25/35/q9]

(a) Express $\dfrac{12x^2+55x-2}{(3x-2)(x+6)}$ in partial fractions. [5]

(b) Hence obtain the expansion of $\dfrac{12x^2+55x-2}{(3x-2)(x+6)}$ in ascending powers of x, up to and including the term in x^2. [4]

5. [9709/w25/31/q10]

Let $f(x) = \dfrac{x^3 + 2x - 11}{(3+x)\left(2+x^2\right)}$.

(a) Express f(x) in partial fractions. [6]

(b) Hence obtain the expansion of f(x) in ascending powers of x, up to and including the term in x^2. [5]

6. [9709/w25/32/q8]

(a) Prove the identity $\sin 4x \equiv 4\sin x\left(2\cos^3 x - \cos x\right)$. [3]

(b) Hence find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^3 x \sin 4x \, dx$. [5]

7. [9709/m24/32/q2]

(a) Find the coefficient of x^2 in the expansion of $(2x-5)\sqrt{4-x}$. [4]

(b) State the set of values of x for which the expansion in part **(a)** is valid. [1]

8. [9709/m24/32/q10]

Let $f(x) = \dfrac{36a^2}{(2a+x)(2a-x)(5a-2x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_{-a}^{a} f(x)\,dx$, giving your answer in the form $p\ln q + r\ln s$ where p and r are integers and q and s are prime numbers. [5]

9. [9709/s24/31/q1]

Expand $(3+x)(1-2x)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

10. [9709/s24/32/q2]

Express $\dfrac{6x^2-9x-16}{2x^2-5x-12}$ in partial fractions. [5]

11. [9709/s24/33/q5]

Express $\frac{6x^2-2x+2}{(x-1)(2x+1)}$ in partial fractions. [5]

12. [9709/w24/31/q7]

Let $f(x) = \dfrac{5x^2 + 8x + 5}{(1 + 2x)(2 + x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the coefficient of x^3 in the expansion of $f(x)$. [4]

13. [9709/w24/32/q1]

Expand $(9-3x)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

14. [9709/w24/33/q8]

Let $f(x) = \dfrac{7a^2}{(a-2x)(3a+x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [4]

(c) State the set of values of x for which the expansion in part **(b)** is valid. [1]

15. [9709/m23/32/q11]

Let $f(x) = \dfrac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\displaystyle\int_0^2 f(x)\,dx = \ln 54 - \tfrac{1}{8}\pi$. [5]

16. [9709/s23/31/q3]

Find the coefficient of x^3 in the binomial expansion of $(3 + x)\sqrt{1 + 4x}$. [4]

17. [9709/s23/31/q8]

Let $f(x) = \dfrac{3 - 3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are integers. [5]

18. [9709/s23/32/q9]

Let $f(x) = \dfrac{2x^2 + 17x - 17}{(1 + 2x)(2 - x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. [5]

19. [9709/s23/33/q10]

Let $f(x) = \dfrac{21 - 8x - 2x^2}{(1 + 2x)(3 - x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

20. [9709/w23/31/q10]

Let $f(x) = \dfrac{24x + 13}{(1 - 2x)(2 + x)^2}$.

(a) Express f(x) in partial fractions. [5]

(b) Hence obtain the expansion of f(x) in ascending powers of x, up to and including the term in x^2. [5]

(c) State the set of values of x for which the expansion in **(b)** is valid. [1]

21. [9709/w23/33/q9]

Let $f(x) = \dfrac{17x^2 - 7x + 16}{(2 + 3x^2)(2 - x)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

(c) State the set of values of x for which the expansion in **(b)** is valid. Give your answer in an exact form. [1]

22. [9709/s22/31/q2]

(a) Expand $(2 - x^2)^{-2}$ in ascending powers of x, up to and including the term in x^4, simplifying the coefficients. [4]

(b) State the set of values of x for which the expansion is valid. [1]

23. [9709/s22/32/q8.a]

Let $\mathrm{f}(x) = \dfrac{x^2 + 9x}{(3x - 1)(x^2 + 3)}$.

(a) Express $\mathrm{f}(x)$ in partial fractions. [5]

(b) Hence find $\displaystyle\int_1^3 \mathrm{f}(x)\,\mathrm{d}x$, giving your answer in a simplified exact form. [5]

24. [9709/s22/33/q7]

Let $f(x) = \dfrac{5x^2 + 8x - 3}{(x-2)(2x^2+3)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

25. [9709/w22/31/q10]

Let $f(x) = \dfrac{2x^2 + 7x + 8}{(1 + x)(2 + x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

26. [9709/w22/32/q10.a]

Let $f(x) = \dfrac{4 - x + x^2}{(1 + x)(2 + x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Find the exact value of $\int_0^4 f(x)\,dx$. Give your answer as a single logarithm. [5]

27. [9709/w22/33/q2]

Expand $\sqrt{\dfrac{1+2x}{1-2x}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

28. [9709/w22/33/q11.a]

Let $f(x) = \dfrac{5 - x + 6x^2}{(3 - x)(1 + 3x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Find the exact value of $\int_0^1 f(x)\,dx$, simplifying your answer. [5]

29. [9709/m21/32/q6.a]

Let $f(x) = \dfrac{5a}{(2x - a)(3a - x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence show that $\displaystyle\int_a^{2a} f(x)\,dx = \ln 6$. [4]

30. [9709/s21/32/q9]

Let $f(x) = \dfrac{14 - 3x + 2x^2}{(2 + x)(3 + x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

31. [9709/s21/33/q1]

Expand $(1 + 3x)^{\frac{2}{3}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

32. [9709/s21/33/q4.a]

Let $f(x) = \dfrac{15 - 6x}{(1 + 2x)(4 - x)}$.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence find $\int_1^2 f(x)\,dx$, giving your answer in the form $\ln\left(\dfrac{a}{b}\right)$, where a and b are integers. [4]

33. [9709/w21/31/q6]

When $(a + bx)\sqrt{1 + 4x}$, where a and b are constants, is expanded in ascending powers of x, the coefficients of x and x^2 are 3 and -6 respectively.

Find the values of a and b. [6]

34. [9709/w21/32/q4]

Express $\dfrac{4x^2 - 13x + 13}{(2x-1)(x-3)}$ in partial fractions. [5]

35. [9709/m20/32/q9]

Let $\mathrm{f}(x) = \dfrac{2 + 11x - 10x^2}{(1 + 2x)(1 - 2x)(2 + x)}$.

(a) Express $\mathrm{f}(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

36. [9709/s20/31/q2]

(a) Expand $(2 - 3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

(b) State the set of values of x for which the expansion is valid. [1]

37. [9709/s20/33/q7.ab]

Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

(a) Express $f(x)$ in partial fractions. [2]

(b) Using your answer to part **(a)**, show that

$$\big(f(x)\big)^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

(c) Hence show that $\int_1^2 \big(f(x)\big)^2\,dx = \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$. [5]

38. [9709/w20/31/q9]

Let $f(x) = \dfrac{8 + 5x + 12x^2}{(1 - x)(2 + 3x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

39. [9709/w20/32/q2]

(a) Expand $\sqrt[3]{1+6x}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

(b) State the set of values of x for which the expansion is valid. [1]

40. [9709/w20/32/q9.a]

Let $f(x) = \dfrac{7x + 18}{(3x + 2)(x^2 + 4)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\displaystyle\int_0^2 f(x)\,dx$. [6]

41. [9709/m19/32/q8]

Let $f(x) = \dfrac{12 + 12x - 4x^2}{(2 + x)(3 - 2x)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

42. [9709/s19/31/q8]

Let $f(x) = \dfrac{16 - 17x}{(2 + x)(3 - x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

43. [9709/s19/32/q1]

Find the coefficient of x^3 in the expansion of $(3 - x)(1 + 3x)^{\frac{1}{3}}$ in ascending powers of x. [4]

44. [9709/m18/32/q2]

Expand $\sqrt[4]{(1-4x)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

45. [9709/m18/32/q8.1]

Let $f(x) = \dfrac{5x^2 + x + 27}{(2x+1)(x^2+9)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence find $\int_0^4 f(x)\,dx$, giving your answer in the form $\ln c$, where c is an integer. [5]

46. [9709/s18/31/q9]

Let $f(x) = \dfrac{12x^2 + 4x - 1}{(x-1)(3x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

47. [9709/s18/32/q9]

Let $\mathrm{f}(x) = \dfrac{x - 4x^2}{(3 - x)(2 + x^2)}$.

(i) Express $\mathrm{f}(x)$ in the form $\dfrac{A}{3 - x} + \dfrac{Bx + C}{2 + x^2}$. [4]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^3. [5]

48. [9709/s18/33/q1]

Expand $\frac{4}{\sqrt{(4-3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

49. [9709/m17/32/q9]

Let $f(x) = \dfrac{x(6-x)}{(2+x)(4+x^2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

50. [9709/s17/32/q8]

Let $f(x) = \dfrac{5x^2 - 7x + 4}{(3x+2)(x^2+5)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

51. [9709/w17/32/q8]

Let $f(x) = \dfrac{8x^2 + 9x + 8}{(1 - x)(2x + 3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

52. [9709/m16/32/q9.1]

Let $f(x) = \dfrac{3x^3 + 6x - 8}{x(x^2 + 2)}$.

(i) Express $f(x)$ in the form $A + \dfrac{B}{x} + \dfrac{Cx + D}{x^2 + 2}$. [5]

(ii) Show that $\displaystyle\int_1^2 f(x)\,dx = 3 - \ln 4$. [5]

53. [9709/s16/31/q8]

Let $f(x) = \dfrac{4x^2 + 12}{(x+1)(x-3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

54. [9709/s16/32/q2]

Expand $\frac{1}{\sqrt{(1-2x)}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

55. [9709/s16/33/q10]

Let $f(x) = \dfrac{10x - 2x^2}{(x+3)(x-1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

56. [9709/w16/31/q2]

Expand $(2 - x)(1 + 2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

57. [9709/w16/33/q8]

Let $f(x) = \dfrac{3x^2 + x + 6}{(x+2)(x^2+4)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

58. [9709/s15/31/q3]

Show that, for small values of x^2,

$$\left(1 - 2x^2\right)^{-2} - \left(1 + 6x^2\right)^{\frac{2}{3}} \approx kx^4,$$

where the value of the constant k is to be determined. [6]

59. [9709/s15/32/q8]

Let $f(x) = \dfrac{5x^2 + x + 6}{(3 - 2x)(x^2 + 4)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

60. [9709/s15/33/q2]

Solve the inequality $|x-2| > 2x-3$. [4]

Chapter 4

Logarithmic and exponential functions

1. [9709/m25/32/q1]

Solve the equation

$$\ln\left(1 - e^{-2x}\right) + 3 = 0.$$

Give your final answer correct to 4 decimal places. [3]

2. [9709/s25/31/q2]

It is given that $2\ln p + \ln(p-1) - \frac{1}{2}\ln(q+1) = 3$.

Find q in terms of p. [3]

3. [9709/s25/32/q1]

Solve the equation $\dfrac{e^x + 2e^{-x}}{e^x - 3} = 4$. Give your answer correct to 3 decimal places. [5]

4. [9709/s25/33/q2]

Solve the equation $2\ln(2x+3)-\ln(2x+5)=\ln(3x)$. [4]

5. [9709/s25/35/q1]

Solve the equation $3^{4-2x} = 5(6^{x-1})$. Give your answer correct to 3 significant figures. [4]

6. [9709/w25/31/q2]

(a) Show that the equation $\log_4(2x+1) = 2\log_4(3x-1) - 2$ can be written as a quadratic equation in x. [3]

(b) Hence solve the equation $\log_4(2x+1) = 2\log_4(3x-1) - 2$. [2]

7. [9709/w25/32/q2]

Solve the equation $3 \times 2^{x+1} = 4 \times 3^{2x-3}$. Give your answer correct to 3 significant figures. [4]

8. [9709/w25/33/q3]

Solve the equation $2^{3x-4} = \dfrac{3}{5^x}$. Give your answer in the form $\dfrac{\ln m}{\ln n}$, where m and n are integers. [4]

9. [9709/w25/35/q3]

The variables x and y satisfy the equation $Ay = b^x$, where A and b are constants.

(a) Show that the graph of $\ln y$ against x is a straight line. [2]

(b) When $x = 3.4$, $\ln y = 0.86$ and when $x = 5.7$, $\ln y = 2.56$.

Find the value of A and the value of b. Give your answers correct to 2 significant figures. [3]

10. [9709/m24/32/q4]

The positive numbers p and q are such that

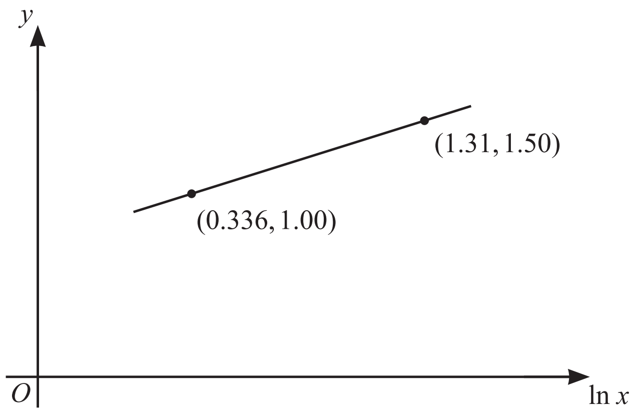
$$\ln\left(\frac{p}{q}\right) = a \quad \text{and} \quad \ln\left(q^2 p\right) = b.$$

Express $\ln\left(p^7 q\right)$ in terms of a and b. [4]

11. [9709/s24/31/q2]

Solve the equation $\ln(x-5) = 7 - \ln x$. Give your answer correct to 2 decimal places. [4]

12. [9709/s24/31/q3]



The variables x and y satisfy the equation $a^y = bx$, where a and b are constants. The graph of y against $\ln x$ is a straight line passing through the points $(0.336, 1.00)$ and $(1.31, 1.50)$, as shown in the diagram.

Find the values of a and b. Give each value correct to the nearest integer. [4]

13. [9709/s24/32/q3]

The variables x and y satisfy the equation $a^{2y-1} = b^{x-y}$, where a and b are constants.

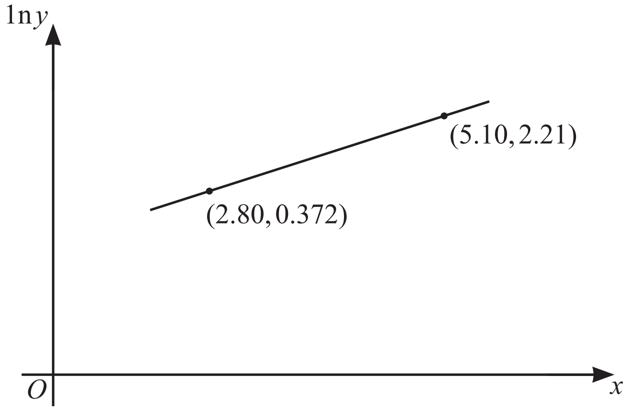
(a) Show that the graph of y against x is a straight line. [3]

(b) Given that $a = b^3$, state the equation of the straight line in the form $y = px + q$, where p and q are rational numbers in their simplest form. [2]

14. [9709/s24/33/q1]

Solve the equation $8^{3-6x} = 4 \times 5^{-2x}$. Give your answer correct to 3 decimal places. [4]

15. [9709/s24/33/q4]



The variables x and y satisfy the equation $ky = e^{cx}$, where k and c are constants. The graph of $\ln y$ against x is a straight line passing through the points $(2.80, 0.372)$ and $(5.10, 2.21)$, as shown in the diagram.

Find the values of k and c. Give each value correct to 2 significant figures. [4]

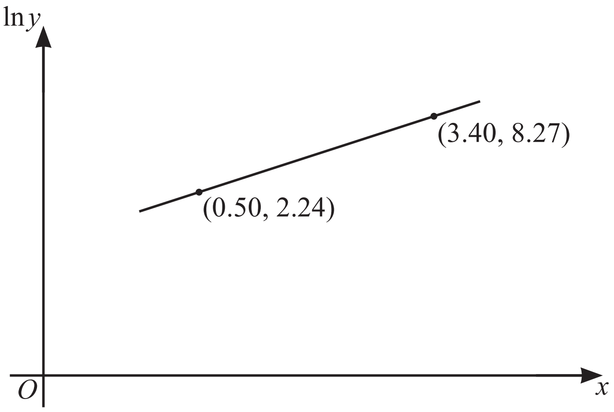
16. [9709/w24/31/q2]

Find the exact value of $\int_1^3 x^2 \ln 3x \, dx$. Give your answer in the form $a \ln b + c$, where a and c are rational and b is an integer. [5]

17. [9709/w24/32/q4]

Solve the equation $5^x = 5^{x+2} - 10$. Give your answer correct to 3 decimal places. [3]

18. [9709/w24/32/q6]



The variables x and y satisfy the equation $ay = b^x$, where a and b are constants. The graph of $\ln y$ against x is a straight line passing through the points (0.50, 2.24) and (3.40, 8.27), as shown in the diagram.

Find the values of a and b. Give each value correct to 1 significant figure. [4]

19. [9709/m23/32/q1]

It is given that $x = \ln(2y - 3) - \ln(y + 4)$.

Express y in terms of x. [3]

20. [9709/s23/31/q1]

Solve the equation

$$3e^{2x} - 4e^{-2x} = 5.$$

Give the answer correct to 3 decimal places. [3]

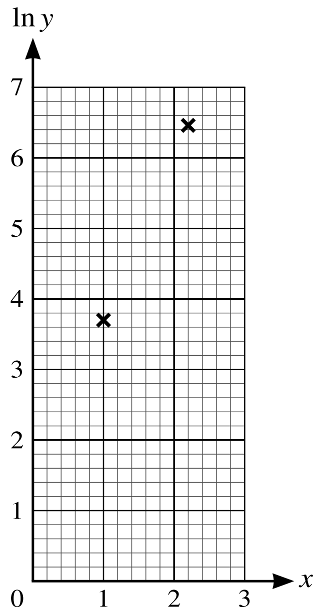
21. [9709/s23/32/q2]

Solve the equation $\ln(2x^2 - 3) = 2\ln x - \ln 2$, giving your answer in an exact form. [3]

22. [9709/s23/33/q1]

Solve the equation $\ln(x+5) = 5 + \ln x$. Give your answer correct to 3 decimal places. [4]

23. [9709/w23/31/q3]



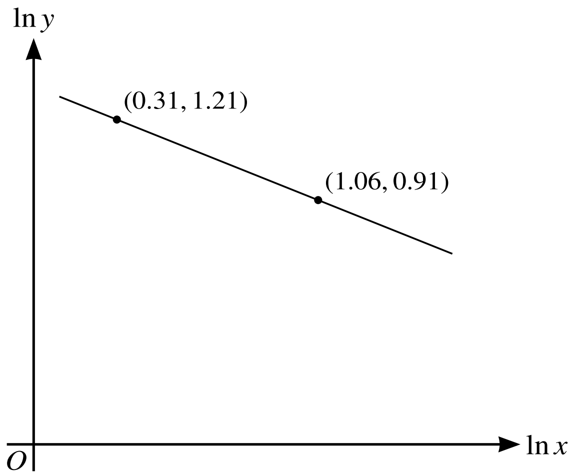
The variables x and y are related by the equation $y = ab^x$, where a and b are constants. The diagram shows the result of plotting $\ln y$ against x for two pairs of values of x and y. The coordinates of these points are (1, 3.7) and (2.2, 6.46).

Use this information to find the values of a and b. [4]

24. [9709/w23/33/q1]

Find the set of values of x satisfying the inequality $\left|2^{x+1} - 2\right| < 0.5$, giving your answer to 3 significant figures. [4]

25. [9709/m22/32/q3]



The variables x and y satisfy the equation $x^n y^2 = C$, where n and C are constants. The graph of $\ln y$ against $\ln x$ is a straight line passing through the points (0.31, 1.21) and (1.06, 0.91), as shown in the diagram.

Find the value of n and find the value of C correct to 2 decimal places. [5]

26. [9709/s22/31/q1]

Solve the equation $2(3^{2x-1}) = 4^{x+1}$, giving your answer correct to 2 decimal places. [4]

27. [9709/s22/32/q1]

Solve the equation $\ln(e^{2x} + 3) = 2x + \ln 3$, giving your answer correct to 3 decimal places. [4]

28. [9709/s22/33/q3]

(a) Show that the equation $\log_3(2x+1) = 1 + 2\log_3(x-1)$ can be written as a quadratic equation in x. [3]

(b) Hence solve the equation $\log_3(4y+1) = 1 + 2\log_3(2y-1)$, giving your answer correct to 2 decimal places. [2]

29. [9709/w22/31/q3]

Solve the equation $2^{3x-1} = 5(3^{-x})$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [4]

30. [9709/w22/32/q1]

Solve the equation $2^{3x-1} = 5(3^{1-x})$. Give your answer in the form $\dfrac{\ln a}{\ln b}$ where a and b are integers. [4]

31. [9709/w22/33/q1]

Solve the equation $\ln(2x - 1) = 2\ln(x + 1) - \ln x$. Give your answer correct to 3 decimal places. [4]

32. [9709/m21/32/q1]

Solve the equation $\ln(x^3 - 3) = 3\ln x - \ln 3$. Give your answer correct to 3 significant figures. [3]

33. [9709/s21/31/q2]

Find the real root of the equation $\frac{2e^x + e^{-x}}{2 + e^x} = 3$, giving your answer correct to 3 decimal places. Your working should show clearly that the equation has only one real root. [5]

34. [9709/s21/32/q3]

The variables x and y satisfy the equation $x = A(3^{-y})$, where A is a constant.

(a) Explain why the graph of y against $\ln x$ is a straight line and state the exact value of the gradient of the line. [3]

It is given that the line intersects the y-axis at the point where $y = 1.3$.

(b) Calculate the value of A, giving your answer correct to 2 decimal places. [2]

35. [9709/s21/33/q2]

Solve the equation $4^x = 3 + 4^{-x}$. Give your answer correct to 3 decimal places. [5]

36. [9709/w21/31/q1]

Solve the equation $4|5^x - 1| = 5^x$, giving your answers correct to 3 decimal places. [4]

37. [9709/w21/32/q1]

Find the value of x for which $3(2^{1-x}) = 7^x$. Give your answer in the form $\dfrac{\ln a}{\ln b}$, where a and b are integers. [4]

38. [9709/w21/33/q3]

Solve the equation $4^{x-2} = 4^x - 4^2$, giving your answer correct to 3 decimal places. [4]

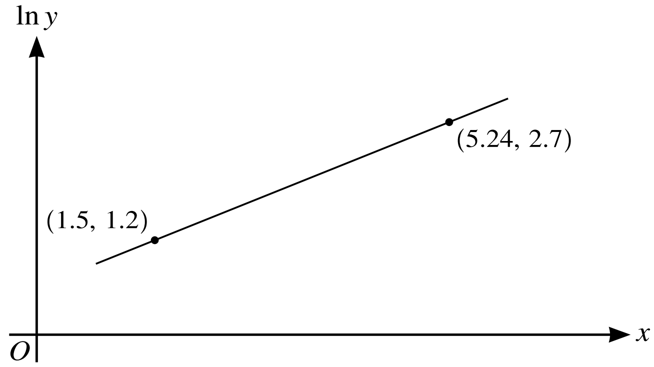
39. [9709/m20/32/q2]

Solve the equation $\ln 3 + \ln(2x + 5) = 2\ln(x + 2)$. Give your answer in a simplified exact form. [4]

40. [9709/s20/31/q1]

Find the set of values of x for which $2(3^{1-2x}) < 5^x$. Give your answer in a simplified exact form. [4]

41. [9709/s20/32/q2]



The variables x and y satisfy the equation $y^2 = Ae^{kx}$, where A and k are constants. The graph of $\ln y$ against x is a straight line passing through the points (1.5, 1.2) and (5.24, 2.7) as shown in the diagram.

Find the values of A and k correct to 2 decimal places. [5]

42. [9709/s20/33/q3]

(a) Show that the equation

$$\ln(1 + e^{-x}) + 2x = 0$$

can be expressed as a quadratic equation in e^x. [2]

(b) Hence solve the equation $\ln(1 + e^{-x}) + 2x = 0$, giving your answer correct to 3 decimal places. [4]

43. [9709/w20/31/q4]

Solve the equation

$$\log_{10}(2x+1) = 2\log_{10}(x+1) - 1.$$

Give your answers correct to 3 decimal places. [6]

44. [9709/w20/32/q1]

Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

45. [9709/w20/32/q3]

The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

46. [9709/m19/32/q1]

(i) Show that the equation $\log_{10}(x-4) = 2 - \log_{10} x$ can be written as a quadratic equation in x. [3]

(ii) Hence solve the equation $\log_{10}(x-4) = 2 - \log_{10} x$, giving your answer correct to 3 significant figures. [2]

47. [9709/s19/31/q2]

Showing all necessary working, solve the equation $\ln(2x - 3) = 2\ln x - \ln(x - 1)$. Give your answer correct to 2 decimal places. [4]

48. [9709/s19/32/q2]

Showing all necessary working, solve the equation $9^x = 3^x + 12$. Give your answer correct to 2 decimal places. [4]

49. [9709/s19/33/q1]

Use logarithms to solve the equation $5^{3-2x} = 4(7^x)$, giving your answer correct to 3 decimal places.

[4]

50. [9709/w19/31/q1]

Given that $\ln\left(1 + e^{2y}\right) = x$, express y in terms of x. [3]

51. [9709/w19/32/q1]

Solve the equation $5\ln(4 - 3^x) = 6$. Show all necessary working and give the answer correct to 3 decimal places. [3]

52. [9709/w19/33/q3]

Showing all necessary working, solve the equation $\frac{3^{2x} + 3^{-x}}{3^{2x} - 3^{-x}} = 4$. Give your answer correct to 3 decimal places. [4]

53. [9709/m18/32/q4]

The variables x and y satisfy the equation $y^n = Ax^3$, where n and A are constants. It is given that $y = 2.58$ when $x = 1.20$, and $y = 9.49$ when $x = 2.51$.

(i) Explain why the graph of $\ln y$ against $\ln x$ is a straight line. [2]

(ii) Find the values of n and A, giving your answers correct to 2 decimal places. [4]

54. [9709/s18/31/q1]

Showing all necessary working, solve the equation $\ln(x^4 - 4) = 4\ln x - \ln 4$, giving your answer correct to 2 decimal places. [4]

55. [9709/s18/33/q2]

Showing all necessary working, solve the equation $5^{2x} = 5^x + 5$. Give your answer correct to 3 decimal places. [5]

56. [9709/w18/31/q2]

Showing all necessary working, solve the equation $\frac{2e^x + e^{-x}}{e^x - e^{-x}} = 4$, giving your answer correct to 2 decimal places. [4]

57. [9709/w18/32/q4]

Showing all necessary working, solve the equation

$$\frac{e^x + e^{-x}}{e^x + 1} = 4,$$

giving your answer correct to 3 decimal places. [5]

58. [9709/m17/32/q1]

Solve the equation $\ln\left(1 + 2^x\right) = 2$, giving your answer correct to 3 decimal places. [3]

59. [9709/s17/32/q1]

Solve the equation $\ln(x^2 + 1) = 1 + 2\ln x$, giving your answer correct to 3 significant figures. [3]

60. [9709/s17/33/q3]

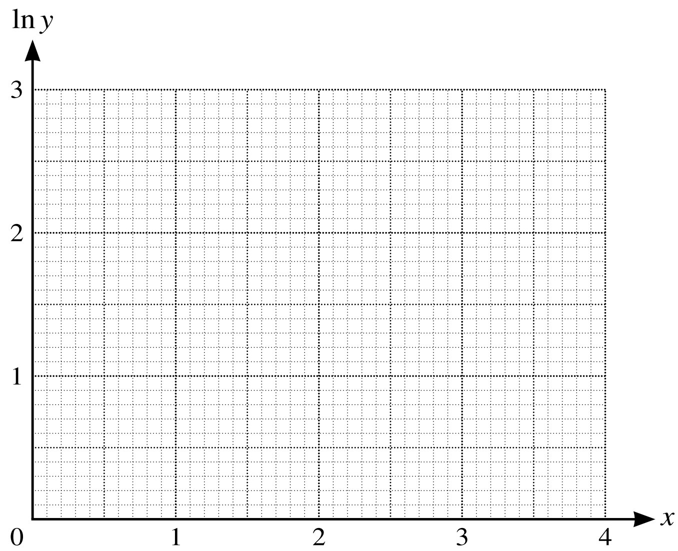
Using the substitution $u = e^x$, solve the equation $4e^{-x} = 3e^x + 4$. Give your answer correct to 3 significant figures. [4]

61. [9709/w17/31/q2]

Two variable quantities x and y are believed to satisfy an equation of the form $y = C(a^x)$, where C and a are constants. An experiment produced four pairs of values of x and y. The table below gives the corresponding values of x and $\ln y$.

x	0.9	1.6	2.4	3.2
$\ln y$	1.7	1.9	2.3	2.6

By plotting $\ln y$ against x for these four pairs of values and drawing a suitable straight line, estimate the values of C and a. Give your answers correct to 2 significant figures. [5]



62. [9709/w17/32/q2]

Showing all necessary working, solve the equation $2\log_2 x = 3 + \log_2(x+1)$, giving your answer correct to 3 significant figures. [5]

63. [9709/m16/32/q1]

Solve the equation $\ln(x^2 + 4) = 2\ln x + \ln 4$, giving your answer in an exact form. [3]

64. [9709/s16/31/q1]

(i) Solve the equation $2|x - 1| = 3|x|$. [3]

(ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

65. [9709/s16/32/q1]

Use logarithms to solve the equation $4^{3x-1} = 3(5^x)$, giving your answer correct to 3 decimal places. [4]

66. [9709/s16/33/q2]

The variables x and y satisfy the relation $3^y = 4^{2-x}$.

(i) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

(ii) Calculate the exact x-coordinate of the point of intersection of this line with the line with equation $y = 2x$, simplifying your answer. [2]

67. [9709/w16/31/q1]

Solve the equation $\dfrac{3^x + 2}{3^x - 2} = 8$, giving your answer correct to 3 decimal places. [3]

68. [9709/w16/33/q1]

It is given that $z = \ln(y+2) - \ln(y+1)$. Express y in terms of z. [3]

69. [9709/s15/31/q1]

Use logarithms to solve the equation $2^{5x} = 3^{2x+1}$, giving the answer correct to 3 significant figures. [4]

70. [9709/s15/32/q2]

Using the substitution $u = 4^x$, solve the equation $4^x + 4^2 = 4^{x+2}$, giving your answer correct to 3 significant figures. [4]

71. [9709/s15/33/q1]

Solve the equation $\ln(x + 4) = 2 \ln x + \ln 4$, giving your answer correct to 3 significant figures. [4]

72. [9709/w15/31/q2]

Using the substitution $u = 3^x$, solve the equation $3^x + 3^{2x} = 3^{3x}$ giving your answer correct to 3 significant figures. [5]

73. [9709/w15/33/q1]

Sketch the graph of $y = e^{ax} - 1$ where a is a positive constant. [2]

Chapter 5

Trigonometry

1. [9709/m25/32/q4]

By first expressing the equation $\tan(x-60°) = 2\cot x$ as a quadratic equation in $\tan x$, solve the equation for $0° \leqslant x \leqslant 180°$. [6]

2. [9709/s25/31/q7]

(a) Express $5\sin\left(x+\frac{1}{6}\pi\right)-4\cos x$ in the form $R\sin(x-\alpha)$, where $R>0$ and $0<\alpha<\frac{1}{2}\pi$. State the exact value of R and give the value of α correct to 3 decimal places. [4]

(b) Hence solve the equation $5\sin\left(2\theta+\frac{1}{6}\pi\right)-4\cos 2\theta=\sqrt{7}$ for $0\leqslant\theta\leqslant\pi$. Give your answers correct to 2 decimal places. [4]

3. [9709/s25/32/q4]

Solve the equation $3\cot x - 4\cot 2x = 3$ for $0° \leqslant x \leqslant 180°$. [6]

4. [9709/s25/32/q7]

(a) Express $7\sin\theta + 24\cos\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to 4 decimal places. [3]

(b) Hence solve the equation $7\sin\frac{1}{3}x + 24\cos\frac{1}{3}x = 24.5$ for $0 < x < \pi$. [4]

5. [9709/s25/33/q8]

(a) Prove the identity $\cot^2\theta - \tan^2\theta \equiv 4\cot 2\theta \operatorname{cosec} 2\theta$. [4]

(b) Hence solve the equation $\cot^2 x - \tan^2 x = 5\sec 2x$ for $0° < x < 90°$. [4]

6. [9709/s25/35/q2]

Solve the equation $3\cot\theta - 4\operatorname{cosec}^2\theta + 5 = 0$ for $-\pi \leqslant \theta \leqslant \pi$. [5]

7. [9709/w25/31/q3]

(a) Express $3\sqrt{2}\sin(x+45°)+\cos x$ in the form $R\cos(x-\alpha)$, where $R>0$ and $0° < \alpha < 90°$. [4]

(b) Hence solve the equation $3\sqrt{2}\sin(3\theta+45°)+\cos 3\theta = -4$ for $0° < \theta < 180°$. [4]

8. [9709/w25/33/q5]

(a) Show that $\cos 4x + 2\sin^2 x - 1 \equiv 8\sin^4 x - 6\sin^2 x$. [4]

(b) Hence solve the equation $\cos 4x + 2\sin^2 x - 1 = 0$ for $-180° \leqslant x \leqslant 180°$. [4]

9. [9709/w25/35/q8]

(a) Express $3\sqrt{3}\sin\left(\theta+\frac{1}{6}\pi\right)-2\sin\theta$ in the form $R\sin(\theta+\alpha)$, where $R>0$ and $0<\alpha<\frac{1}{2}\pi$. Give the exact value of R and state the value of α correct to 3 decimal places. [5]

(b) Hence, solve the equation $3\sqrt{3}\sin\left(2x+\frac{1}{6}\pi\right)-2\sin 2x=\sqrt{6}$ for $0<x<\pi$. [4]

10. [9709/m24/32/q8]

(a) Express $3\sin x + 2\sqrt{2}\cos\left(x + \frac{1}{4}\pi\right)$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. State the exact value of R and give α correct to 3 decimal places. [4]

(b) Hence solve the equation

$$6\sin\tfrac{1}{2}\theta + 4\sqrt{2}\cos\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi\right) = 3$$

for $-4\pi < \theta < 4\pi$. [5]

11. [9709/s24/32/q7]

(a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$. [3]

(b) Hence find the exact value of $\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \left(\cos^4\theta - \sin^4\theta + 4\sin^2\theta\cos^2\theta\right) \mathrm{d}\theta$. [6]

12. [9709/s24/33/q8]

(a) Express $3\cos 2x - \sqrt{3}\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and α. [3]

(b) Hence find the exact value of $\int_0^{\frac{1}{12}\pi} \frac{3}{\left(3\cos 2x - \sqrt{3}\sin 2x\right)^2}\,dx$, simplifying your answer. [5]

13. [9709/w24/31/q4]

(a) Show that $\sec^4\theta - \tan^4\theta \equiv 1 + 2\tan^2\theta$. [3]

(b) Hence or otherwise solve the equation $\sec^4 2\alpha - \tan^4 2\alpha = 2\tan^2 2\alpha \sec^2 2\alpha$ for $0° < \alpha < 180°$. [5]

14. [9709/w24/32/q7]

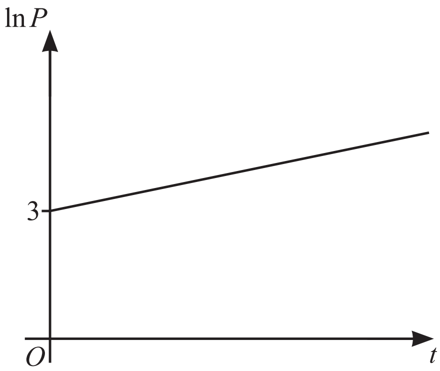
(a) Show that the equation $\tan^3 x + 2\tan 2x - \tan x = 0$ may be expressed as

$$\tan^4 x - 2\tan^2 x - 3 = 0$$

for $\tan x \neq 0$. [3]

(b) Hence solve the equation $\tan^3 2\theta + 2\tan 4\theta - \tan 2\theta = 0$ for $0 < \theta < \pi$. Give your answers in exact form. [3]

15. [9709/w24/33/q3]



The number of bacteria in a population, P, at time t hours is modelled by the equation $P = ae^{kt}$, where a and k are constants. The graph of $\ln P$ against t, shown in the diagram, has gradient $\frac{1}{20}$ and intersects the vertical axis at $(0, 3)$.

(a) State the value of k and find the value of a correct to 2 significant figures. [3]

(b) Find the time taken for P to double. Give your answer correct to the nearest hour. [2]

16. [9709/w24/33/q5]

(a) Show that $\cos^4\theta - \sin^4\theta - 4\sin^2\theta\cos^2\theta \equiv \cos^2 2\theta + \cos 2\theta - 1$. [3]

(b) Solve the equation $\cos^4\alpha - \sin^4\alpha = 4\sin^2\alpha\cos^2\alpha$ for $0° \leqslant \alpha \leqslant 180°$. [3]

17. [9709/m23/32/q6]

(a) Express $5\sin\theta + 12\cos\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. [3]

(b) Hence solve the equation $5\sin 2x + 12\cos 2x = 6$ for $0 \leqslant x \leqslant \pi$. [4]

18. [9709/s23/31/q4]

(a) Show that the equation $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$ can be expressed in the form

$$\cos^2\theta + 2\sin\theta\cos\theta - 3\sin^2\theta = 0.$$ [2]

(b) Hence solve the equation $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$ for $0° < \theta < 180°$. [4]

19. [9709/s23/32/q4]

Solve the equation $2\cos x - \cos\frac{1}{2}x = 1$ for $0 \leqslant x \leqslant 2\pi$. [5]

20. [9709/s23/33/q6]

(a) Express $3\cos x + 2\cos(x - 60°)$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [4]

(b) Hence solve the equation

$$3\cos 2\theta + 2\cos(2\theta - 60°) = 2.5$$

for $0° < \theta < 180°$. [4]

21. [9709/w23/31/q5]

(a) Given that

$$\sin(x + \tfrac{1}{6}\pi) - \sin(x - \tfrac{1}{6}\pi) = \cos(x + \tfrac{1}{3}\pi) - \cos(x - \tfrac{1}{3}\pi),$$

find the exact value of $\tan x$. [4]

(b) Hence find the exact roots of the equation

$$\sin(x + \tfrac{1}{6}\pi) - \sin(x - \tfrac{1}{6}\pi) = \cos(x + \tfrac{1}{3}\pi) - \cos(x - \tfrac{1}{3}\pi)$$

for $0 \leqslant x \leqslant 2\pi$. [2]

22. [9709/w23/32/q7]

(a) By expressing 3θ as $2\theta + \theta$, prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [3]

(b) Hence solve the equation

$$\cos 3\theta + \cos\theta\cos 2\theta = \cos^2\theta$$

for $0° \leqslant \theta \leqslant 180°$. [5]

23. [9709/w23/33/q6]

(a) Show that the equation $\cot^2\theta + 2\cos 2\theta = 4$ can be written in the form

$$4\sin^4\theta + 3\sin^2\theta - 1 = 0.$$ [3]

(b) Hence solve the equation $\cot^2\theta + 2\cos 2\theta = 4$, for $0° < \theta < 360°$. [3]

24. [9709/m22/32/q5]

The angles α and β lie between $0°$ and $180°$ and are such that

$$\tan(\alpha + \beta) = 2 \quad \text{and} \quad \tan\alpha = 3\tan\beta.$$

Find the possible values of α and β. [6]

25. [9709/s22/31/q3]

Solve the equation $2\cot 2x + 3\cot x = 5$, for $0° < x < 180°$. [6]

26. [9709/s22/32/q2]

Solve the equation $3\cos 2\theta = 3\cos\theta + 2$, for $0° \leqslant \theta \leqslant 360°$. [5]

27. [9709/s22/33/q2]

Solve the equation $\cos(\theta - 60^\circ) = 3\sin\theta$, for $0^\circ \leqslant \theta \leqslant 360^\circ$. [5]

28. [9709/w22/31/q4]

Solve the equation $\tan(x + 45°) = 2\cot x$ for $0° < x < 180°$. [5]

29. [9709/w22/31/q6]

(a) Prove the identity $\cos 4\theta + 4\cos 2\theta + 3 \equiv 8\cos^4\theta$. [4]

(b) Hence solve the equation $\cos 4\theta + 4\cos 2\theta = 4$ for $0° \leqslant \theta \leqslant 180°$. [3]

30. [9709/w22/32/q4]

(a) Express $4\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [3]

(b) Hence solve the equation $4\cos 2x - \sin 2x = 3$ for $0° < x < 180°$. [5]

31. [9709/w22/33/q7]

(a) Show that the equation $\sqrt{5}\sec x + \tan x = 4$ can be expressed as $R\cos(x + \alpha) = \sqrt{5}$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [4]

(b) Hence solve the equation $\sqrt{5}\sec 2x + \tan 2x = 4$, for $0° < x < 180°$. [4]

32. [9709/m21/32/q3]

By first expressing the equation $\tan(x + 45°) = 2\cot x + 1$ as a quadratic equation in $\tan x$, solve the equation for $0° < x < 180°$. [6]

33. [9709/m21/32/q5]

(a) Express $\sqrt{7}\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [3]

(b) Hence solve the equation $\sqrt{7}\sin 2\theta + 2\cos 2\theta = 1$, for $0° < \theta < 180°$. [5]

34. [9709/s21/31/q3]

(a) Given that $\cos(x - 30°) = 2\sin(x + 30°)$, show that $\tan x = \dfrac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$. [4]

(b) Hence solve the equation

$$\cos(x - 30°) = 2\sin(x + 30°),$$

for $0° < x < 360°$. [2]

35. [9709/s21/32/q6.a]

(a) Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$. [3]

(b) Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\,d\theta = \frac{1}{2}\ln 2$. [4]

36. [9709/s21/33/q5]

(a) By first expanding $\tan(2\theta + 2\theta)$, show that the equation $\tan 4\theta = \frac{1}{2}\tan\theta$ may be expressed as $\tan^4\theta + 2\tan^2\theta - 7 = 0$. [4]

(b) Hence solve the equation $\tan 4\theta = \frac{1}{2}\tan\theta$, for $0° < \theta < 180°$. [3]

37. [9709/w21/31/q2]

(a) Express $5\sin x - 3\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact value of R and give α correct to 2 decimal places. [3]

(b) Hence state the greatest and least possible values of $(5\sin x - 3\cos x)^2$. [2]

38. [9709/w21/31/q5]

(a) Show that the equation

$$\cot 2\theta + \cot \theta = 2$$

can be expressed as a quadratic equation in $\tan \theta$. [3]

(b) Hence solve the equation $\cot 2\theta + \cot \theta = 2$, for $0 < \theta < \pi$, giving your answers correct to 3 decimal places. [3]

39. [9709/w21/32/q8]

(a) By first expanding $\left(\cos^2\theta + \sin^2\theta\right)^2$, show that

$$\cos^4\theta + \sin^4\theta \equiv 1 - \tfrac{1}{2}\sin^2 2\theta.$$ [3]

(b) Hence solve the equation

$$\cos^4\theta + \sin^4\theta = \tfrac{5}{9},$$

for $0° < \theta < 180°$. [4]

40. [9709/w21/33/q5]

Solve the equation $\sin\theta = 3\cos 2\theta + 2$, for $0° \leqslant \theta \leqslant 360°$. [5]

41. [9709/w21/33/q6]

(a) By first expanding $\cos(x - 60°)$, show that the expression

$$2\cos(x - 60°) + \cos x$$

can be written in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [5]

(b) Hence find the value of x in the interval $0° < x < 360°$ for which $2\cos(x - 60°) + \cos x$ takes its least possible value. [2]

42. [9709/m20/32/q5]

(a) Show that $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 2\cot 2x$. [4]

(b) Hence solve the equation $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 4$, for $0 < x < \pi$. [3]

43. [9709/s20/31/q3]

Express the equation $\tan(\theta + 60^\circ) = 2 + \tan(60^\circ - \theta)$ as a quadratic equation in $\tan\theta$, and hence solve the equation for $0^\circ \leqslant \theta \leqslant 180^\circ$. [6]

44. [9709/s20/32/q5]

(a) Express $\sqrt{2}\cos x - \sqrt{5}\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 3 decimal places. [3]

(b) Hence solve the equation $\sqrt{2}\cos 2\theta - \sqrt{5}\sin 2\theta = 1$, for $0° < \theta < 180°$. [4]

45. [9709/s20/33/q5]

By first expressing the equation

$$\tan\theta\tan(\theta+45^\circ) = 2\cot 2\theta$$

as a quadratic equation in $\tan\theta$, solve the equation for $0^\circ < \theta < 90^\circ$. [6]

46. [9709/w20/31/q6]

(a) Express $\sqrt{6}\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [3]

(b) Hence solve the equation $\sqrt{6}\cos\frac{1}{3}x + 3\sin\frac{1}{3}x = 2.5$, for $0° < x < 360°$. [4]

47. [9709/w20/32/q4]

(a) Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0.$$ [3]

(b) Hence solve the equation $\tan(\theta + 60°) = 2\cot\theta$, for $0° < \theta < 180°$. [3]

48. [9709/m19/32/q3]

(i) Given that $\sin(\theta + 45°) + 2\cos(\theta + 60°) = 3\cos\theta$, find the exact value of $\tan\theta$ in a form involving surds. You need not simplify your answer. [4]

(ii) Hence solve the equation $\sin(\theta + 45°) + 2\cos(\theta + 60°) = 3\cos\theta$ for $0° < \theta < 360°$. [2]

49. [9709/s19/31/q4]

By first expressing the equation $\cot\theta - \cot(\theta + 45°) = 3$ as a quadratic equation in $\tan\theta$, solve the equation for $0° < \theta < 180°$. [6]

50. [9709/s19/32/q3]

Showing all necessary working, solve the equation $\cot 2\theta = 2\tan\theta$ for $0° < \theta < 180°$. [5]

51. [9709/w19/32/q4]

(i) Express $(\sqrt{6})\sin x + \cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 3 decimal places. [3]

(ii) Hence solve the equation $(\sqrt{6})\sin 2\theta + \cos 2\theta = 2$, for $0° < \theta < 180°$. [4]

52. [9709/w19/33/q4]

(i) By first expanding $\tan(2x + x)$, show that the equation $\tan 3x = 3 \cot x$ can be written in the form $\tan^4 x - 12 \tan^2 x + 3 = 0$. [4]

(ii) Hence solve the equation $\tan 3x = 3 \cot x$ for $0° < x < 90°$. [3]

53. [9709/m18/32/q3.1]

(i) Using the expansions of $\cos(3x + x)$ and $\cos(3x - x)$, show that

$$\tfrac{1}{2}(\cos 4x + \cos 2x) \equiv \cos 3x \cos x. \quad [3]$$

(ii) Hence show that $\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x \cos x \, dx = \frac{3}{8}\sqrt{3}$. [3]

54. [9709/s18/31/q2]

(i) Given that $\sin(x - 60°) = 3\cos(x - 45°)$, find the exact value of $\tan x$. [4]

(ii) Hence solve the equation $\sin(x - 60°) = 3\cos(x - 45°)$, for $0° < x < 360°$. [2]

55. [9709/s18/32/q2]

Showing all necessary working, solve the equation $\cot\theta + \cot(\theta + 45°) = 2$, for $0° < \theta < 180°$. [5]

56. [9709/s18/33/q5]

(i) By first expanding $\left(\cos^2 x + \sin^2 x\right)^3$, or otherwise, show that

$$\cos^6 x + \sin^6 x = 1 - \tfrac{3}{4}\sin^2 2x. \qquad [4]$$

(ii) Hence solve the equation

$$\cos^6 x + \sin^6 x = \tfrac{2}{3},$$

for $0° < x < 180°$. [4]

57. [9709/w18/31/q6]

(i) Show that the equation $(\sqrt{2})\operatorname{cosec} x + \cot x = \sqrt{3}$ can be expressed in the form $R\sin(x - \alpha) = \sqrt{2}$, where $R > 0$ and $0° < \alpha < 90°$. [4]

(ii) Hence solve the equation $(\sqrt{2})\operatorname{cosec} x + \cot x = \sqrt{3}$, for $0° < x < 180°$. [4]

58. [9709/w18/32/q2]

Showing all necessary working, solve the equation $\sin(\theta - 30°) + \cos\theta = 2\sin\theta$, for $0° < \theta < 180°$. [4]

59. [9709/m17/32/q4]

(i) Express $8\cos\theta - 15\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the exact value of R and giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$8\cos 2x - 15\sin 2x = 4,$$

for $0° < x < 180°$. [4]

60. [9709/s17/31/q8]

(i) By first expanding $2\sin(x - 30°)$, express $2\sin(x - 30°) - \cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$2\sin(x - 30°) - \cos x = 1,$$

for $0° < x < 180°$. [3]

61. [9709/s17/32/q3]

(i) Express the equation $\cot\theta - 2\tan\theta = \sin 2\theta$ in the form $a\cos^4\theta + b\cos^2\theta + c = 0$, where a, b and c are constants to be determined. [3]

(ii) Hence solve the equation $\cot\theta - 2\tan\theta = \sin 2\theta$ for $90° < \theta < 180°$. [2]

62. [9709/s17/33/q1]

Prove the identity $\dfrac{\cot x - \tan x}{\cot x + \tan x} \equiv \cos 2x$. [3]

63. [9709/w17/31/q4]

(i) Prove the identity $\tan(45° + x) + \tan(45° - x) \equiv 2 \sec 2x$. [4]

(ii) Hence sketch the graph of $y = \tan(45° + x) + \tan(45° - x)$ for $0° \leqslant x \leqslant 90°$. [3]

64. [9709/w17/32/q3]

By expressing the equation $\tan(\theta+60°)+\tan(\theta-60°)=\cot\theta$ in terms of $\tan\theta$ only, solve the equation for $0° < \theta < 90°$. [5]

65. [9709/m16/32/q2]

Express the equation $\tan(\theta + 45°) - 2\tan(\theta - 45°) = 4$ as a quadratic equation in $\tan\theta$. Hence solve this equation for $0° \leqslant \theta \leqslant 180°$. [6]

66. [9709/s16/31/q3]

By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. [5]

67. [9709/s16/32/q5]

(i) Prove the identity $\cos 4\theta - 4\cos 2\theta \equiv 8\sin^4\theta - 3$. [4]

(ii) Hence solve the equation

$$\cos 4\theta = 4\cos 2\theta + 3,$$

for $0° \leqslant \theta \leqslant 360°$. [4]

68. [9709/s16/33/q3]

(i) Express $(\sqrt{5})\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$(\sqrt{5})\cos\tfrac{1}{2}x + 2\sin\tfrac{1}{2}x = 1.2,$$

for $0° < x < 360°$. [3]

69. [9709/w16/31/q3]

Express the equation $\sec\theta = 3\cos\theta + \tan\theta$ as a quadratic equation in $\sin\theta$. Hence solve this equation for $-90^\circ < \theta < 90^\circ$. [5]

70. [9709/w16/33/q3]

Express the equation $\cot 2\theta = 1 + \tan\theta$ as a quadratic equation in $\tan\theta$. Hence solve this equation for $0° < \theta < 180°$. [6]

71. [9709/s15/32/q4]

(i) Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the exact value of R and giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$3\sin\theta + 2\cos\theta = 1,$$

for $0° < \theta < 180°$. [3]

72. [9709/s15/33/q3]

Solve the equation $\cot 2x + \cot x = 3$ for $0° < x < 180°$. [6]

73. [9709/w15/31/q3]

The angles θ and ϕ lie between 0° and 180°, and are such that

$$\tan(\theta - \phi) = 3 \quad \text{and} \quad \tan\theta + \tan\phi = 1.$$

Find the possible values of θ and ϕ. [6]

74. [9709/w15/33/q6]

The angles A and B are such that

$$\sin(A + 45°) = (2\sqrt{2})\cos A \quad \text{and} \quad 4\sec^2 B + 5 = 12\tan B.$$

Without using a calculator, find the exact value of $\tan(A - B)$. [8]

Chapter 6

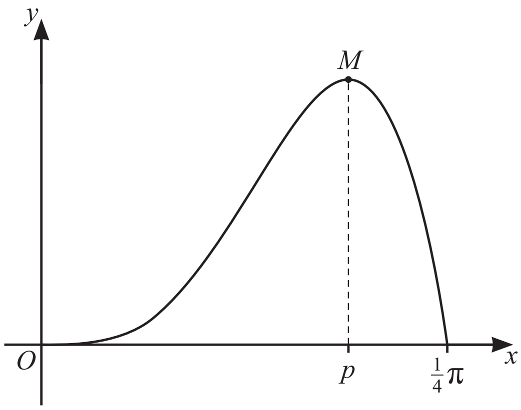
Differentiation

1. [9709/m25/32/q2]

The equation of a curve is $xy^2 + \ln(x + 2y) = 1$.

Find the gradient of the curve at the point where $x = 0$. [5]

2. [9709/m25/32/q7]



The diagram shows the curve $y = x^3 \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The curve has a maximum point at M, where $x = p$.

(a) Show that p satisfies the equation $p = \frac{1}{2}\tan^{-1}\left(\frac{3}{2p}\right)$. [3]

(b) Show by calculation that $0.5 < p < 0.7$. [2]

(c) Use an iterative formula based on the equation in part **(a)** to calculate p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

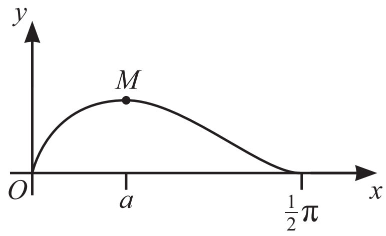
3. [9709/s25/31/q4]

The parametric equations of a curve are

$$x = e^{\tan t}, \quad y = 3\tan^2 t.$$

Find the equation of the tangent to the curve at the point (e, 3). Give your answer in the form $y = mx + c$, where m and c are exact. [6]

4. [9709/s25/31/q11]



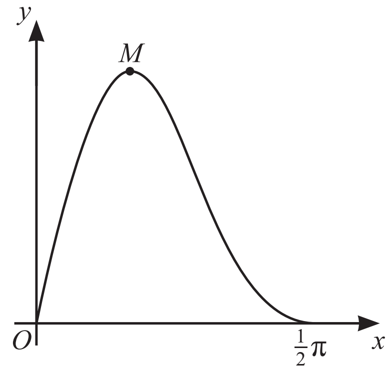
The diagram shows the curve $y = \cos x\sqrt{\sin 2x}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curve has a maximum point at M, where $x = a$.

(a) Find the exact value of a. [6]

(b) The region enclosed between the x-axis and the curve is rotated through 2π radians about the x-axis.

Find the exact volume of the solid generated. [5]

5. [9709/s25/32/q11]



The diagram shows the graph of $y = 5\sin 2x\cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and its maximum point M.

(a) Find the exact x-coordinate of M. [6]

(b) By using the substitution $u = \cos x$, find the area of the region bounded by the curve, the x-axis between $x = 0$ and $x = \frac{1}{4}\pi$, and the line $x = \frac{1}{4}\pi$. [5]

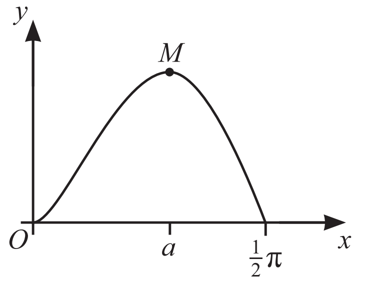
6. [9709/s25/33/q5]

The equation of a curve is $xy + y^2 e^{-x} = 4$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y^2 - ye^x}{xe^x + 2y}$. [4]

(b) Find the gradients of the tangents to the curve when $x = 0$. [2]

7. [9709/s25/33/q11]



The diagram shows the curve $y = \sqrt{x}\sin 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curve has a maximum point at M, where $x = a$.

(a) Show that $\tan 2a = -4a$ [4]

(b) Show by calculation that 0 9 $< a < 0.95$. [2]

(c) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\left(\pi - \tan^{-1}\left(4x_n\right)\right)$$

converges, then it converges to a. [2]

(d) Use the iterative formula in part **(c)** to calculate a correct to 4 decimal places. Give the result of each iteration to 6 decimal places. [3]

8. [9709/s25/35/q4]

Find the exact coordinates of the stationary point of the curve with equation $y = 3x^3 \ln x^4$, for $x > 0$. [5]

9. [9709/s25/35/q6]

The parametric equations of a curve are

$$x = \frac{2}{\cos 3t} \qquad \text{and} \qquad y = \tan 3t,$$

for $0 \leqslant t \leqslant 2\pi$.

(a) Show that $\frac{dy}{dx}$ can be written as $A \operatorname{cosec} 3t$, where A is a constant to be found. [5]

(b) Find an equation of the normal to the curve at the point where $t = \frac{1}{12}\pi$. Give your answer in the form $y = mx + c$, where the constants m and c are exact. [4]

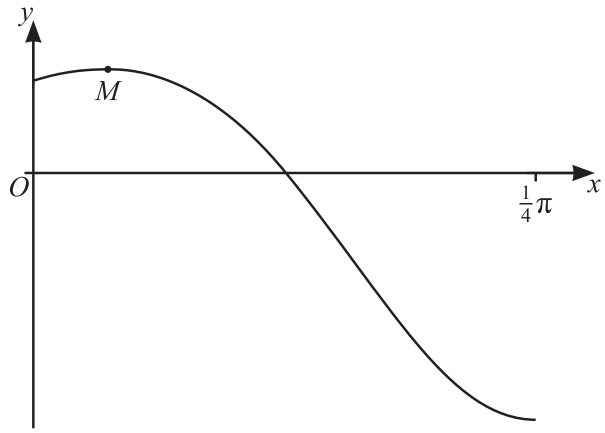
10. [9709/s25/35/q7]

The equation of a curve is $y = \tan^{-1}(4x)$.

(a) Find the exact values of x when the gradient of the curve is $\frac{1}{4}$. [3]

(b) Find the exact value of $\int_0^{0.25} y\,dx$. [5]

11. [9709/w25/31/q4]



The diagram shows the graph of $y = e^{\sin 2x} \cos 4x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$, and its maximum point M.

Find the x-coordinate of M. [5]

12. [9709/w25/31/q7]

The parametric equations of a curve are

$$x = t^2 - \ln(2t+1), \quad y = \frac{t}{2t+1}.$$

Obtain a simplified expression for $\frac{dy}{dx}$ in terms of t. [5]

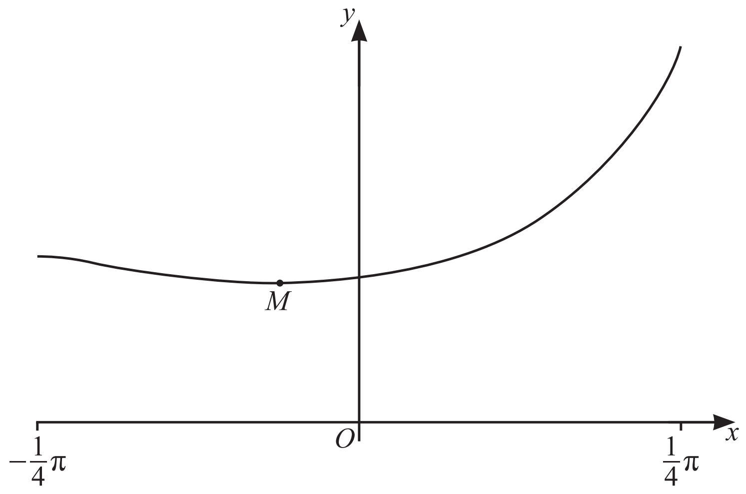
13. [9709/w25/32/q7]

The equation of a curve is $2y^3 - 3x^2y - x^3 = 16$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x^2 + 2xy}{2y^2 - x^2}$. [4]

(b) Hence find the coordinates of the points on the curve at which the normal is parallel to the y-axis. [4]

14. [9709/w25/33/q11]



The diagram shows the graph of $y = \sec^2 x\sqrt{3+2\tan x}$ for $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$, and its minimum point M.

(a) Find the x-coordinate of M. [6]

(b) Using the substitution $u = 3+2\tan x$, find the exact value of the area of the region bounded by the curve, the x-axis and the lines $x = -\frac{1}{4}\pi$ and $x = \frac{1}{4}\pi$. [6]

15. [9709/w25/35/q4]

The equation of a curve is $x^2 \ln 2y - y \ln\left(2 + x^2\right) = \ln 6$.

Find the exact value of the gradient of the curve at the point (2, 3). Give your answer in simplified form. [5]

16. [9709/w25/35/q6]

A curve has equation $y = \frac{\tan x}{5 + 2\sin x}$.

(a) Express $\frac{dy}{dx}$ as a simplified fraction in terms of $\sin x$. [4]

(b) Show that the curve has no stationary points. [2]

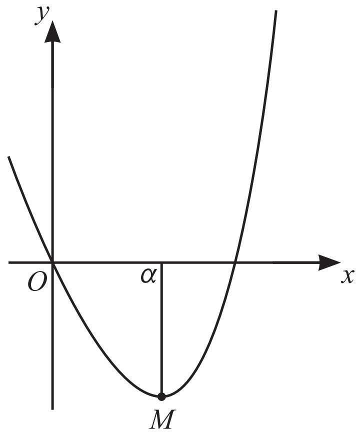
17. [9709/m24/32/q6]

The equation of a curve is $2y^2 + 3xy + x = x^2$.

(a) Show that $\frac{dy}{dx} = \frac{2x - 3y - 1}{4y + 3x}$. [4]

(b) Hence show that the curve does **not** have a tangent that is parallel to the x-axis. [3]

18. [9709/m24/32/q7]



The diagram shows the curve $y = xe^{2x} - 5x$ and its minimum point M, where $x = \alpha$.

(a) Show that α satisfies the equation $\alpha = \frac{1}{2}\ln\left(\frac{5}{1+2\alpha}\right)$. [3]

(b) Verify by calculation that α lies between 0.4 and 0.5 . [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

19. [9709/s24/31/q5]

The equation of a curve is $y = \dfrac{e^{\sin x}}{\cos^2 x}$ for $0 \leqslant x \leqslant 2\pi$.

Find $\dfrac{dy}{dx}$ and hence find the x-coordinates of the stationary points of the curve. [7]

20. [9709/s24/31/q10]

(a) Given that $2x = \tan y$, show that $\dfrac{dy}{dx} = \dfrac{2}{1+4x^2}$. [3]

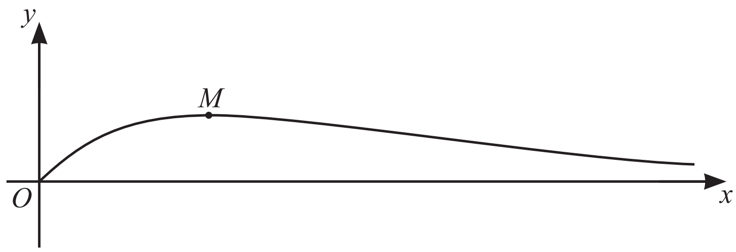
(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x \tan^{-1}(2x)\,dx$. [7]

21. [9709/s24/32/q4]

The equation of a curve is $ye^{2x} + y^2e^x = 6$.

Find the gradient of the curve at the point where $y = 1$. [6]

22. [9709/s24/32/q6]



The diagram shows the curve $y = xe^{-ax}$, where a is a positive constant, and its maximum point M.

(a) Find the exact coordinates of M. [4]

(b) Find the exact value of $\int_0^{\frac{2}{a}} xe^{-ax}\,dx$. [5]

23. [9709/s24/33/q2]

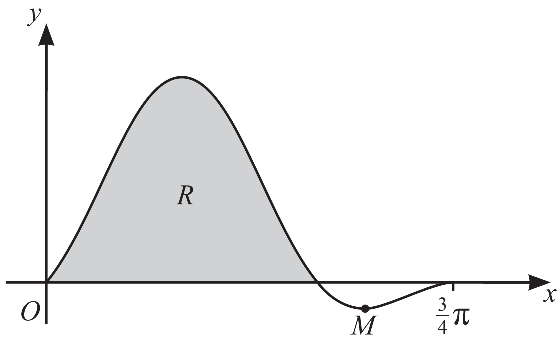
Find the exact coordinates of the stationary point of the curve $y = e^{2x} \sin 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. [5]

24. [9709/w24/31/q3]

The equation of a curve is $\ln(x+y) = 3x^2y$.

Find the gradient of the curve at the point $(1, 0)$. [4]

25. [9709/w24/31/q6]



The diagram shows the curve $y = \sin 2x(1 + \sin 2x)$, for $0 \leqslant x \leqslant \frac{3}{4}\pi$, and its minimum point M. The shaded region bounded by the curve that lies above the x-axis and the x-axis itself is denoted by R.

(a) Given that the x-coordinate of M lies in the interval $\frac{1}{2}\pi < x < \frac{3}{4}\pi$, find the exact coordinates of M. [4]

(b) Find the exact area of the region R. [4]

26. [9709/w24/32/q8]

The parametric equations of a curve are

$$x = \tan^2 2t, \quad y = \cos 2t,$$

for $0 < t < \frac{1}{4}\pi$.

(a) Show that $\frac{dy}{dx} = -\frac{1}{2}\cos^3 2t$. [4]

(b) Hence find the equation of the normal to the curve at the point where $t = \frac{1}{8}\pi$. Give your answer in the form $y = mx + c$. [4]

27. [9709/w24/32/q11]

Let $f(x) = \dfrac{2e^{2x}}{e^{2x} - 3e^{x} + 2}$.

(a) Find $f'(x)$ and hence find the exact coordinates of the stationary point of the curve with equation $y = f(x)$. [5]

(b) Use the substitution $u = e^x$ and partial fractions to find the exact value of $\int_{\ln 3}^{\ln 5} f(x)\,dx$.

Give your answer in the form $\ln a$, where a is a rational number in its simplest form. [9]

28. [9709/w24/33/q7]

The parametric equations of a curve are

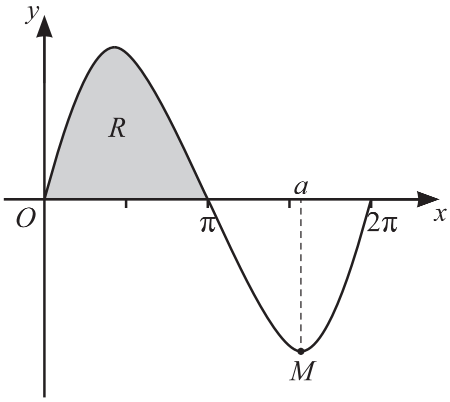
$$x = 3\sin 2t, \qquad y = \tan t + \cot t,$$

for $0 < t < \frac{1}{2}\pi$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-2}{3\sin^2 2t}$. [5]

(b) Find the equation of the normal to the curve at the point where $t = \frac{1}{4}\pi$. Give your answer in the form $py + qx + r = 0$, where p, q and r are integers. [3]

29. [9709/w24/33/q11]



The diagram shows the curve $y = 2\sin x\sqrt{2+\cos x}$, for $0 \leqslant x \leqslant 2\pi$, and its minimum point M, where $x = a$.

(a) Find the value of a correct to 2 decimal places. [5]

(b) Use the substitution $u = 2 + \cos x$ to find the exact area of the shaded region R. [6]

30. [9709/m23/32/q5]

The parametric equations of a curve are

$$x = te^{2t}, \qquad y = t^2 + t + 3.$$

(a) Show that $\frac{dy}{dx} = e^{-2t}$. [3]

(b) Hence show that the normal to the curve, where $t = -1$, passes through the point $\left(0,\ 3 - \frac{1}{e^4}\right)$. [3]

31. [9709/s23/31/q5]

The equation of a curve is $x^2y - ay^2 = 4a^3$, where a is a non-zero constant.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2xy}{2ay - x^2}$. [4]

(b) Hence find the coordinates of the points where the tangent to the curve is parallel to the y-axis. [4]

32. [9709/s23/32/q7]

The equation of a curve is $3x^2 + 4xy + 3y^2 = 5$.

(a) Show that $\frac{dy}{dx} = -\frac{3x + 2y}{2x + 3y}$. [4]

(b) Hence find the exact coordinates of the two points on the curve at which the tangent is parallel to $y + 2x = 0$. [5]

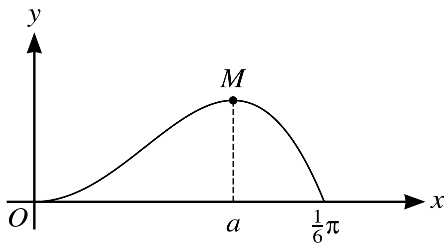
33. [9709/s23/33/q4]

The parametric equations of a curve are

$$x = \frac{\cos\theta}{2 - \sin\theta}, \qquad y = \theta + 2\cos\theta.$$

Show that $\frac{dy}{dx} = (2 - \sin\theta)^2$. [5]

34. [9709/s23/33/q5]



The diagram shows the part of the curve $y = x^2 \cos 3x$ for $0 \leqslant x \leqslant \frac{1}{6}\pi$, and its maximum point M, where $x = a$.

(a) Show that a satisfies the equation $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$. [3]

(b) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

35. [9709/w23/31/q1]

Find the exact coordinates of the points on the curve $y = \dfrac{x^2}{1 - 3x}$ at which the gradient of the tangent is equal to 8. [5]

36. [9709/w23/31/q6]

The parametric equations of a curve are

$$x = \sqrt{t} + 3, \qquad y = \ln t,$$

for $t > 0$.

(a) Obtain a simplified expression for $\frac{dy}{dx}$ in terms of t. [3]

(b) Hence find the exact coordinates of the point on the curve at which the gradient of the normal is -2. [3]

37. [9709/w23/32/q2]

The parametric equations of a curve are

$$x = (\ln t)^2, \qquad y = e^{2-t^2},$$

for $t > 0$.

Find the gradient of the curve at the point where $t = e$, simplifying your answer. [4]

38. [9709/w23/33/q5]

Find the exact coordinates of the stationary points of the curve $y = \dfrac{e^{3x^2-1}}{1-x^2}$. [6]

39. [9709/w23/33/q7]

The equation of a curve is $x^3 + y^2 + 3x^2 + 3y = 4$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 6x}{2y + 3}$. [3]

(b) Hence find the coordinates of the points on the curve at which the tangent is parallel to the x-axis. [5]

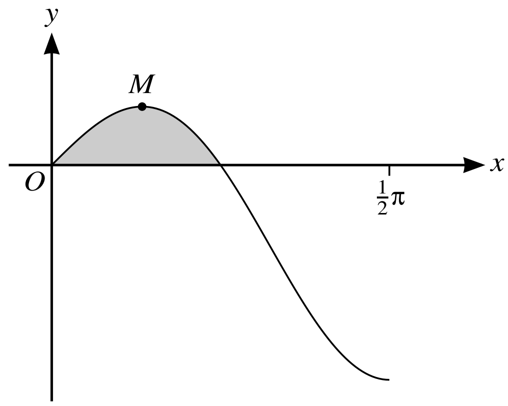
40. [9709/m22/32/q4]

The parametric equations of a curve are

$$x = 1 - \cos\theta, \qquad y = \cos\theta - \tfrac{1}{4}\cos 2\theta.$$

Show that $\dfrac{dy}{dx} = -2\sin^2\left(\tfrac{1}{2}\theta\right)$. [5]

41. [9709/m22/32/q11.a]



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 significant figures. [6]

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. [5]

42. [9709/s22/31/q8]

The equation of a curve is $x^3 + y^3 + 2xy + 8 = 0$.

(a) Express $\frac{dy}{dx}$ in terms of x and y. [4]

The tangent to the curve at the point where $x = 0$ and the tangent at the point where $y = 0$ intersect at the acute angle α.

(b) Find the exact value of $\tan \alpha$. [5]

43. [9709/s22/32/q4]

The equation of a curve is $y = \cos^3 x\sqrt{\sin x}$. It is given that the curve has one stationary point in the interval $0 < x < \frac{1}{2}\pi$.

Find the x-coordinate of this stationary point, giving your answer correct to 3 significant figures. [6]

44. [9709/s22/32/q7]

The equation of a curve is $x^3 + 3x^2y - y^3 = 3$.

(a) Show that $\frac{dy}{dx} = \frac{x^2 + 2xy}{y^2 - x^2}$. [4]

(b) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

45. [9709/s22/33/q4]

The curve $y = e^{-4x} \tan x$ has two stationary points in the interval $0 \leqslant x < \frac{1}{2}\pi$.

(a) Obtain an expression for $\frac{dy}{dx}$ and show it can be written in the form $\sec^2 x(a + b \sin 2x)e^{-4x}$, where a and b are constants. [4]

(b) Hence find the exact x-coordinates of the two stationary points. [3]

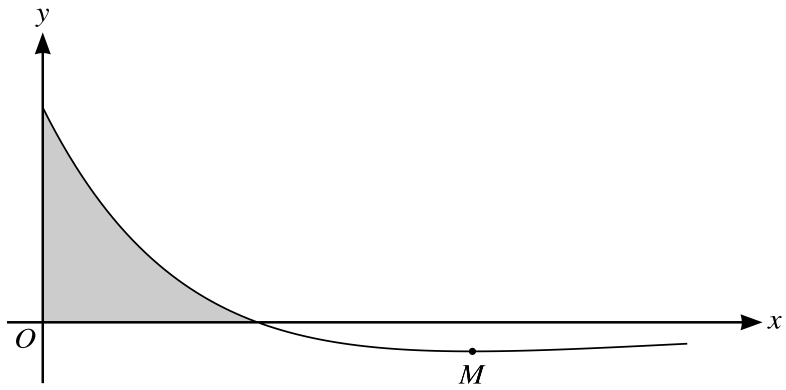
46. [9709/s22/33/q6]

The parametric equations of a curve are $x = \dfrac{1}{\cos t}$, $y = \ln \tan t$, where $0 < t < \frac{1}{2}\pi$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{\cos t}{\sin^2 t}$. [5]

(b) Find the equation of the tangent to the curve at the point where $y = 0$. [3]

47. [9709/w22/31/q9.a]



The diagram shows part of the curve $y = (3 - x)e^{-\frac{1}{3}x}$ for $x \geqslant 0$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

(b) Find the area of the shaded region bounded by the curve and the axes, giving your answer in terms of e. [5]

48. [9709/w22/32/q3]

The equation of a curve is $y = \sin x \sin 2x$. The curve has a stationary point in the interval $0 < x < \frac{1}{2}\pi$.

Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

49. [9709/w22/33/q4]

The parametric equations of a curve are

$$x = 2t - \tan t, \qquad y = \ln(\sin 2t),$$

for $0 < t < \frac{1}{2}\pi$.

Show that $\frac{dy}{dx} = \cot t$. [5]

50. [9709/m21/32/q9.c]

Let $f(x) = \dfrac{e^{2x} + 1}{e^{2x} - 1}$, for $x > 0$.

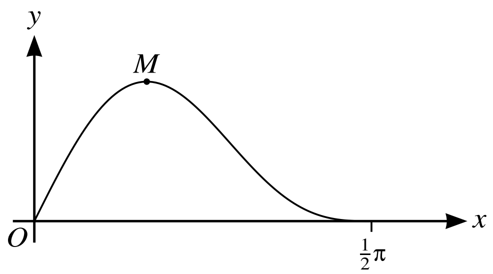
(a) The equation $x = f(x)$ has one root, denoted by a.

Verify by calculation that a lies between 1 and 1.5. [2]

(b) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(c) Find $f'(x)$. Hence find the exact value of x for which $f'(x) = -8$. [6]

51. [9709/m21/32/q10.b]



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. [5]

(b) Find the exact x-coordinate of M. [6]

52. [9709/s21/31/q6]

The parametric equations of a curve are

$$x = \ln(2 + 3t), \qquad y = \frac{t}{2 + 3t}.$$

(a) Show that the gradient of the curve is always positive. [5]

(b) Find the equation of the tangent to the curve at the point where it intersects the y-axis. [3]

53. [9709/s21/31/q9.a]

The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point. [5]

(b) Show that $\int_1^8 y \, dx = 18 \ln 2 - 9$. [5]

54. [9709/s21/32/q8]

The equation of a curve is $y = e^{-5x} \tan^2 x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

Find the x-coordinates of the stationary points of the curve. Give your answers correct to 3 decimal places where appropriate. [8]

55. [9709/s21/33/q3]

The parametric equations of a curve are

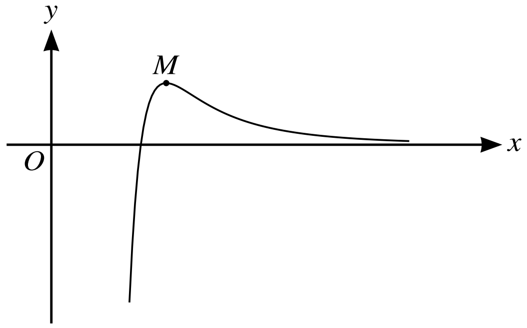
$$x = t + \ln(t + 2), \qquad y = (t - 1)e^{-2t},$$

where $t > -2$.

(a) Express $\dfrac{dy}{dx}$ in terms of t, simplifying your answer. [5]

(b) Find the exact y-coordinate of the stationary point of the curve. [2]

56. [9709/s21/33/q8.a]



The diagram shows the curve $y = \dfrac{\ln x}{x^4}$ and its maximum point M.

(a) Find the exact coordinates of M. [4]

(b) By using integration by parts, show that for all $a > 1$, $\displaystyle\int_1^a \frac{\ln x}{x^4}\,dx < \frac{1}{9}$. [6]

57. [9709/w21/31/q3]

The curve with equation $y = xe^{1-2x}$ has one stationary point.

(a) Find the coordinates of this point. [4]

(b) Determine whether the stationary point is a maximum or a minimum. [2]

58. [9709/w21/32/q9]

The equation of a curve is $ye^{2x} - y^2e^x = 2$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2ye^x - y^2}{2y - e^x}$. [4]

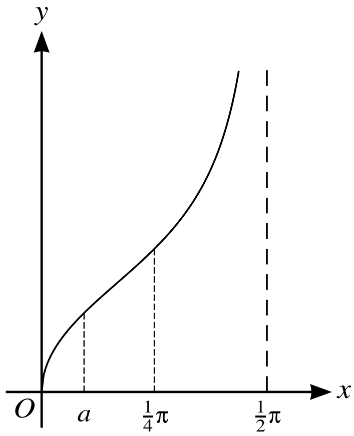
(b) Find the exact coordinates of the point on the curve where the tangent is parallel to the y-axis. [4]

59. [9709/w21/32/q11.ab]

The equation of a curve is $y = \sqrt{\tan x}$, for $0 \leqslant x < \frac{1}{2}\pi$.

(a) Express $\frac{dy}{dx}$ in terms of $\tan x$, and verify that $\frac{dy}{dx} = 1$ when $x = \frac{1}{4}\pi$. [4]

The value of $\frac{dy}{dx}$ is also 1 at another point on the curve where $x = a$, as shown in the diagram.



(b) Show that $t^3 + t^2 + 3t - 1 = 0$, where $t = \tan a$. [4]

(c) Use the iterative formula

$$a_{n+1} = \tan^{-1}\left(\tfrac{1}{3}(1 - \tan^2 a_n - \tan^3 a_n)\right)$$

to determine a correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

60. [9709/w21/33/q7]

The equation of a curve is $\ln(x + y) = x - 2y$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x + y - 1}{2(x + y) + 1}$. [4]

(b) Find the coordinates of the point on the curve where the tangent is parallel to the x-axis. [3]

61. [9709/w21/33/q9.a]

Let $f(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

(a) Find the x-coordinate of the stationary point of the curve with equation $y = f(x)$. [4]

(b) Using the substitution $u = \sqrt{x}$, show that $\int_0^4 f(x)\,dx = \frac{1}{3}\ln 5$. [6]

62. [9709/m20/32/q7]

The equation of a curve is $x^3 + 3xy^2 - y^3 = 5$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{y^2 - 2xy}$. [4]

(b) Find the coordinates of the points on the curve where the tangent is parallel to the y-axis. [5]

63. [9709/s20/31/q4]

The curve with equation $y = e^{2x}(\sin x + 3\cos x)$ has a stationary point in the interval $0 \leqslant x \leqslant \pi$.

(a) Find the x-coordinate of this point, giving your answer correct to 2 decimal places. [4]

(b) Determine whether the stationary point is a maximum or a minimum. [2]

64. [9709/s20/31/q7.a]

Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$. [4]

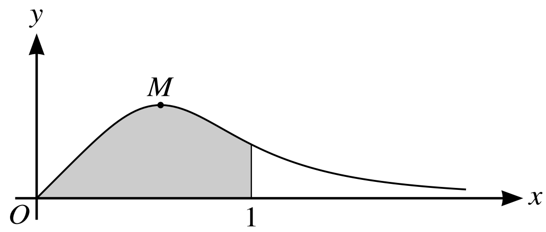
(b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

65. [9709/s20/32/q4]

A curve has equation $y = \cos x \sin 2x$.

Find the x-coordinate of the stationary point in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

66. [9709/s20/32/q6.a]



The diagram shows the curve $y = \dfrac{x}{1 + 3x^4}$, for $x \geqslant 0$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [4]

(b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [5]

67. [9709/s20/33/q4]

The equation of a curve is $y = x\tan^{-1}\left(\frac{1}{2}x\right)$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [3]

(b) The tangent to the curve at the point where $x = 2$ meets the y-axis at the point with coordinates $(0, p)$.

Find p. [3]

68. [9709/w20/31/q3]

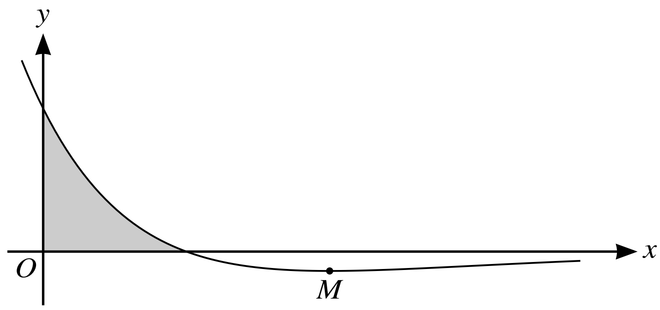
The parametric equations of a curve are

$$x = 3 - \cos 2\theta, \quad y = 2\theta + \sin 2\theta,$$

for $0 < \theta < \frac{1}{2}\pi$.

Show that $\frac{dy}{dx} = \cot\theta$. [5]

69. [9709/w20/31/q10.a]

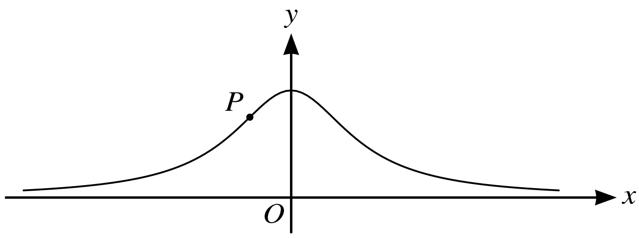


The diagram shows the curve $y = (2 - x)e^{-\frac{1}{2}x}$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

(b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e. [5]

70. [9709/w20/32/q5]



The diagram shows the curve with parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta,$$

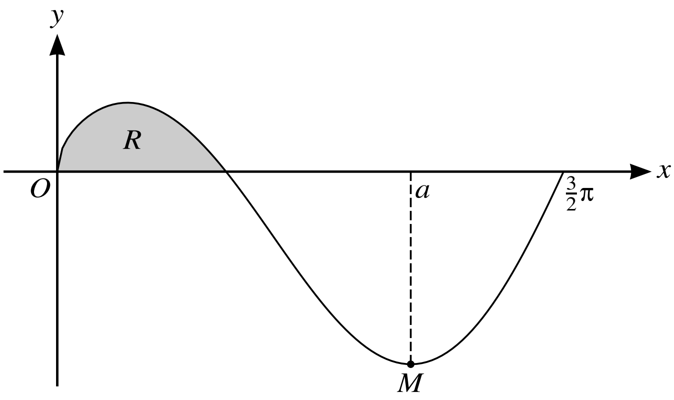
for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$. [3]

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. [4]

71. [9709/w20/32/q10.a]



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \dfrac{1}{2a}$. [3]

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

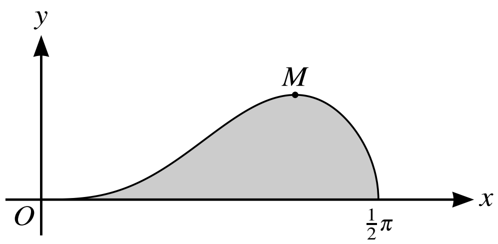
72. [9709/m19/32/q5]

The variables x and y satisfy the relation $\sin y = \tan x$, where $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$. Show that

$$\frac{dy}{dx} = \frac{1}{\cos x \sqrt{(\cos 2x)}}.$$

[5]

73. [9709/m19/32/q10.2]



The diagram shows the curve $y = \sin^3 x\sqrt{(\cos x)}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Showing all your working, find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

74. [9709/s19/31/q3]

Find the gradient of the curve $x^3 + 3xy^2 - y^3 = 1$ at the point with coordinates $(1, 3)$. [4]

75. [9709/s19/32/q4]

Find the exact coordinates of the point on the curve $y = \dfrac{x}{1 + \ln x}$ at which the gradient of the tangent is equal to $\frac{1}{4}$. [7]

76. [9709/s19/33/q4]

The equation of a curve is $y = \dfrac{1 + e^{-x}}{1 - e^{-x}}$, for $x > 0$.

(i) Show that $\dfrac{dy}{dx}$ is always negative. [3]

(ii) The gradient of the curve is equal to -1 when $x = a$. Show that a satisfies the equation $e^{2a} - 4e^{a} + 1 = 0$. Hence find the exact value of a. [4]

77. [9709/s19/33/q7]

The curve $y = \sin\left(x + \frac{1}{3}\pi\right)\cos x$ has two stationary points in the interval $0 \leqslant x \leqslant \pi$.

(i) Find $\dfrac{dy}{dx}$. [2]

(ii) By considering the formula for $\cos(A + B)$, show that, at the stationary points on the curve, $\cos\left(2x + \frac{1}{3}\pi\right) = 0$. [2]

(iii) Hence find the exact x-coordinates of the stationary points. [3]

78. [9709/w19/31/q3]

The parametric equations of a curve are

$$x = 2t + \sin 2t, \quad y = \ln(1 - \cos 2t).$$

Show that $\frac{dy}{dx} = \operatorname{cosec} 2t$. [5]

79. [9709/w19/32/q2]

The curve with equation $y = \frac{e^{-2x}}{1 - x^2}$ has a stationary point in the interval $-1 < x < 1$. Find $\frac{dy}{dx}$ and hence find the x-coordinate of this stationary point, giving the answer correct to 3 decimal places. [5]

80. [9709/w19/32/q5]

The equation of a curve is $2x^2y - xy^2 = a^3$, where a is a positive constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis and find the y-coordinate of this point. [7]

81. [9709/m18/32/q5]

The parametric equations of a curve are

$$x = 2t + \sin 2t, \quad y = 1 - 2\cos 2t,$$

for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = 2\tan t$. [5]

(ii) Hence find the x-coordinate of the point on the curve at which the gradient of the normal is 2. Give your answer correct to 3 significant figures. [2]

82. [9709/s18/31/q3]

A curve has equation $y = \dfrac{e^{3x}}{\tan \frac{1}{2}x}$. Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. Give your answers correct to 3 decimal places. [6]

83. [9709/s18/32/q5]

The equation of a curve is $x^2(x + 3y) - y^3 = 3$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 + 2xy}{y^2 - x^2}$. [4]

(ii) Hence find the exact coordinates of the two points on the curve at which the gradient of the normal is 1. [4]

84. [9709/s18/33/q8]

The equation of a curve is $2x^3 - y^3 - 3xy^2 = 2a^3$, where a is a non-zero constant.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2x^2 - y^2}{y^2 + 2xy}$. [4]

(ii) Find the coordinates of the two points on the curve at which the tangent is parallel to the y-axis. [5]

85. [9709/w18/31/q4]

The parametric equations of a curve are

$$x = 2\sin\theta + \sin 2\theta, \quad y = 2\cos\theta + \cos 2\theta,$$

where $0 < \theta < \pi$.

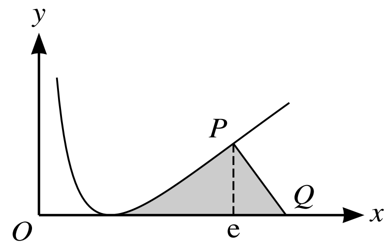
(i) Obtain an expression for $\dfrac{dy}{dx}$ in terms of θ. [3]

(ii) Hence find the exact coordinates of the point on the curve at which the tangent is parallel to the y-axis. [4]

86. [9709/m17/32/q5]

The curve with equation $y = e^{-ax} \tan x$, where a is a positive constant, has only one point in the interval $0 < x < \frac{1}{2}\pi$ at which the tangent is parallel to the x-axis. Find the value of a and state the exact value of the x-coordinate of this point. [7]

87. [9709/m17/32/q10.1]



The diagram shows the curve $y = (\ln x)^2$. The x-coordinate of the point P is equal to e, and the normal to the curve at P meets the x-axis at Q.

(i) Find the x-coordinate of Q. [4]

(ii) Show that $\int \ln x \, dx = x \ln x - x + c$, where c is a constant. [1]

(iii) Using integration by parts, or otherwise, find the exact value of the area of the shaded region between the curve, the x-axis and the normal PQ. [5]

88. [9709/s17/31/q4]

The parametric equations of a curve are

$$x = \ln\cos\theta, \quad y = 3\theta - \tan\theta,$$

where $0 \leqslant \theta < \frac{1}{2}\pi$.

(i) Express $\dfrac{dy}{dx}$ in terms of $\tan\theta$. [5]

(ii) Find the exact y-coordinate of the point on the curve at which the gradient of the normal is equal to 1. [3]

89. [9709/s17/32/q4]

The parametric equations of a curve are

$$x = t^2 + 1, \quad y = 4t + \ln(2t - 1).$$

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Find the equation of the normal to the curve at the point where $t = 1$. Give your answer in the form $ax + by + c = 0$. [3]

90. [9709/s17/33/q5]

A curve has equation $y = \frac{2}{3}\ln(1 + 3\cos^2 x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $\tan x$. [4]

(ii) Hence find the x-coordinate of the point on the curve where the gradient is -1. Give your answer correct to 3 significant figures. [2]

91. [9709/w17/31/q5]

The equation of a curve is $2x^4 + xy^3 + y^4 = 10$.

(i) Show that $\frac{dy}{dx} = -\frac{8x^3 + y^3}{3xy^2 + 4y^3}$. [4]

(ii) Hence show that there are two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

92. [9709/w17/32/q4]

The curve with equation $y = \dfrac{2 - \sin x}{\cos x}$ has one stationary point in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

(i) Find the exact coordinates of this point. [5]

(ii) Determine whether this point is a maximum or a minimum point. [2]

93. [9709/w17/32/q6]

The equation of a curve is $x^3y - 3xy^3 = 2a^4$, where a is a non-zero constant.

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - 3y^3}{9xy^2 - x^3}$. [4]

(ii) Hence show that there are only two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

94. [9709/m16/32/q6]

A curve has equation

$$\sin y \ln x = x - 2\sin y,$$

for $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [5]

(ii) Hence find the exact x-coordinate of the point on the curve at which the tangent is parallel to the x-axis. [3]

95. [9709/s16/31/q5]

The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

96. [9709/s16/31/q7]

The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - y^2}$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

97. [9709/s16/32/q4]

The curve with equation $y = \dfrac{(\ln x)^2}{x}$ has two stationary points. Find the exact values of the coordinates of these points. [6]

98. [9709/s16/33/q4]

The parametric equations of a curve are

$$x = t + \cos t, \qquad y = \ln(1 + \sin t),$$

where $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \sec t$. [5]

(ii) Hence find the x-coordinates of the points on the curve at which the gradient is equal to 3. Give your answers correct to 3 significant figures. [3]

99. [9709/w16/31/q4]

The equation of a curve is $xy(x - 6y) = 9a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [7]

100. [9709/w16/33/q2]

The equation of a curve is $y = \dfrac{\sin x}{1 + \cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all x in the given interval. [4]

101. [9709/s15/31/q4]

The equation of a curve is

$$y = 3\cos 2x + 7\sin x + 2.$$

Find the x-coordinates of the stationary points in the interval $0 \leqslant x \leqslant \pi$. Give each answer correct to 3 significant figures. [7]

102. [9709/s15/32/q3]

A curve has equation $y = \cos x \cos 2x$. Find the x-coordinate of the stationary point on the curve in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

103. [9709/s15/33/q4]

The curve with equation $y = \dfrac{e^{2x}}{4 + e^{3x}}$ has one stationary point. Find the exact values of the coordinates of this point. [6]

104. [9709/s15/33/q5]

The parametric equations of a curve are

$$x = a\cos^4 t, \quad y = a\sin^4 t,$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin^2 t + y\cos^2 t = a\sin^2 t\cos^2 t.$$ [3]

(iii) Hence show that if the tangent meets the x-axis at P and the y-axis at Q, then

$$OP + OQ = a,$$

where O is the origin. [2]

105. [9709/w15/31/q5]

The equation of a curve is $y = e^{-2x} \tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b \tan x)^2$, where a and b are constants. [5]

(ii) Explain why the gradient of the curve is never negative. [1]

(iii) Find the value of x for which the gradient is least. [1]

106. [9709/w15/33/q3]

A curve has equation

$$y = \frac{2 - \tan x}{1 + \tan x}.$$

Find the equation of the tangent to the curve at the point for which $x = \frac{1}{4}\pi$, giving the answer in the form $y = mx + c$ where c is correct to 3 significant figures. [6]

Chapter 7

Integration

1. [9709/m25/32/q10]

Let $f(x) = \dfrac{-7x^2 + 2x - 6}{(1+x)(4+x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. Give your answer in the form $a\pi - \ln b$, where a and b are constants. [6]

2. [9709/m25/32/q11]

Find the exact value of $\int_0^{\pi} x^2 \cos \frac{1}{3}x \, dx$. [6]

3. [9709/s25/31/q9]

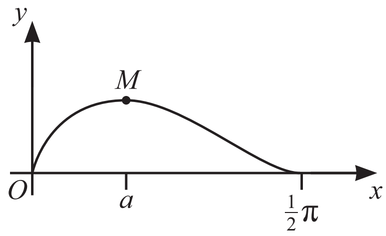
The constant a is such that $\int_1^a 6x \ln x \, dx = 4$.

(a) Show that $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$, where $\exp(x)$ denotes e^x. [5]

(b) Verify by calculation that a lies between 2 and 2.1. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

4. [9709/s25/31/q11]



The diagram shows the curve $y = \cos x\sqrt{\sin 2x}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curve has a maximum point at M, where $x = a$.

(a) Find the exact value of a. [6]

(b) The region enclosed between the x-axis and the curve is rotated through 2π radians about the x-axis.

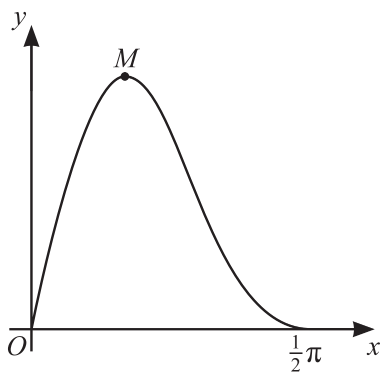
Find the exact volume of the solid generated. [5]

5. [9709/s25/32/q10]

(a) Find the quotient and remainder when x^2 is divided by $1+4x^2$. [2]

(b) Find the exact value of $\int_0^{0.5} x\tan^{-1}(2x)\,dx$. [6]

6. [9709/s25/32/q11]



The diagram shows the graph of $y = 5\sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and its maximum point M.

(a) Find the exact x-coordinate of M. [6]

(b) By using the substitution $u = \cos x$, find the area of the region bounded by the curve, the x-axis between $x = 0$ and $x = \frac{1}{4}\pi$, and the line $x = \frac{1}{4}\pi$. [5]

7. [9709/s25/33/q3]

Find the exact value of $\int_{\frac{1}{5}\pi}^{\frac{1}{4}\pi} 3\cos^2 5x \, dx$. [4]

8. [9709/s25/35/q7]

The equation of a curve is $y = \tan^{-1}(4x)$.

(a) Find the exact values of x when the gradient of the curve is $\frac{1}{4}$. [3]

(b) Find the exact value of $\int_0^{0.25} y\,dx$. [5]

9. [9709/w25/31/q1]

Find the exact value of $\int_1^2 \ln 3x \, dx$. Give your answer in the form $a + \ln b$, where a and b are integers. [4]

10. [9709/w25/32/q4]

Find the exact value of $\int_0^1 x\tan^{-1}x\,\mathrm{d}x$. [6]

11. [9709/w25/32/q8]

(a) Prove the identity $\sin 4x \equiv 4\sin x\left(2\cos^3 x - \cos x\right)$. [3]

(b) Hence find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^3 x \sin 4x \, dx$. [5]

12. [9709/w25/32/q9]

Let $f(x) = \dfrac{x^2 + 4ax + 6a^2}{(x+2a)(x+3a)}$, where a is a positive constant.

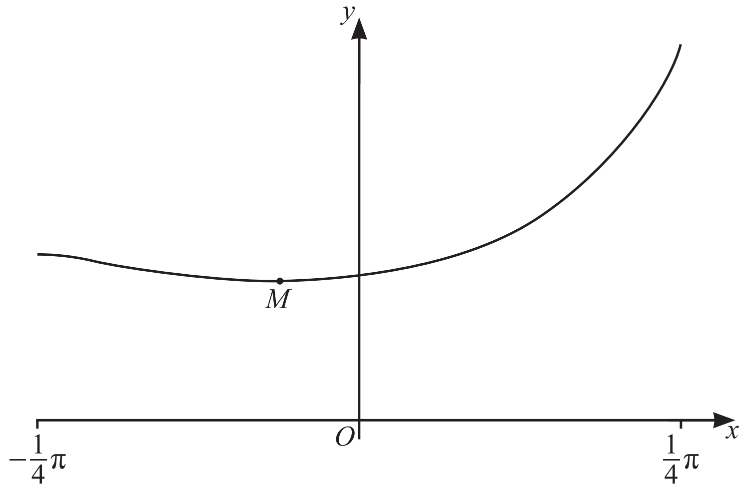
(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_{-a}^{a} f(x)\,dx$. Give your answer in the form $a(p + \ln q)$, where p and q are rational. [4]

13. [9709/w25/33/q6]

Find the exact value of $\int_0^{\frac{1}{6}\pi} x^2 \sin 2x \, dx$. [6]

14. [9709/w25/33/q11]



The diagram shows the graph of $y = \sec^2 x\sqrt{3+2\tan x}$ for $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$, and its minimum point M.

(a) Find the x-coordinate of M. [6]

(b) Using the substitution $u = 3+2\tan x$, find the exact value of the area of the region bounded by the curve, the x-axis and the lines $x = -\frac{1}{4}\pi$ and $x = \frac{1}{4}\pi$. [6]

15. [9709/w25/35/q7]

(a) Use the substitution $u = x^2 - 3$ to show that

$$\int_{\sqrt{7}}^{\sqrt{12}} \frac{4x^3}{\sqrt{x^2-3}}\,dx = \int_a^b \frac{2(u+3)}{\sqrt{u}}\,du,$$

where a and b are values to be found. [4]

(b) Hence, find the exact value of $\displaystyle\int_{\sqrt{7}}^{\sqrt{12}} \frac{4x^3}{\sqrt{x^2-3}}\,dx$. [4]

16. [9709/w25/35/q10]

The constant a is such that $\int_0^a x\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x = 6$.

(a) Show that $a = 2 + \mathrm{e}^{-\frac{1}{2}a}$. [5]

(b) Verify by calculation that a lies between 2.2 and 2.4. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

17. [9709/m24/32/q10]

Let $f(x) = \dfrac{36a^2}{(2a+x)(2a-x)(5a-2x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\displaystyle\int_{-a}^{a} f(x)\,dx$, giving your answer in the form $p\ln q + r\ln s$ where p and r are integers and q and s are prime numbers. [5]

18. [9709/s24/31/q8]

Use the substitution $u = 1 - \sin x$ to find the exact value of

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{\sin 2x}{\sqrt{1-\sin x}}\,\mathrm{d}x.$$

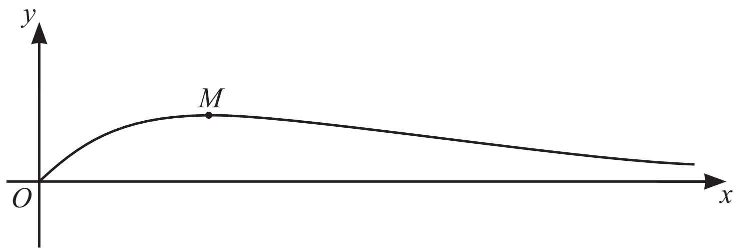
Give your answer in the form $a + b\sqrt{2}$ where a and b are rational numbers to be determined. [7]

19. [9709/s24/31/q10]

(a) Given that $2x = \tan y$, show that $\dfrac{dy}{dx} = \dfrac{2}{1+4x^2}$. [3]

(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x\tan^{-1}(2x)\,dx$. [7]

20. [9709/s24/32/q6]



The diagram shows the curve $y = xe^{-ax}$, where a is a positive constant, and its maximum point M.

(a) Find the exact coordinates of M. [4]

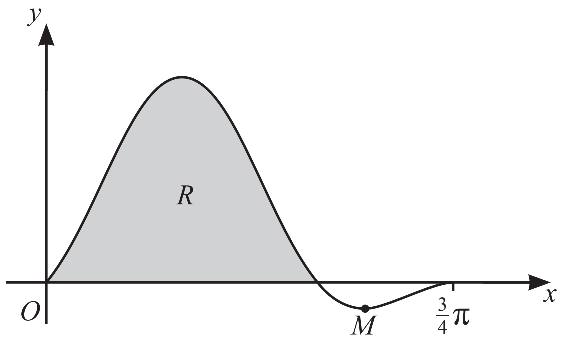
(b) Find the exact value of $\int_0^{\frac{2}{a}} xe^{-ax}\,dx$. [5]

21. [9709/s24/32/q7]

(a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$. [3]

(b) Hence find the exact value of $\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \left(\cos^4\theta - \sin^4\theta + 4\sin^2\theta\cos^2\theta\right) \mathrm{d}\theta$. [6]

22. [9709/w24/31/q6]



The diagram shows the curve $y = \sin 2x(1 + \sin 2x)$, for $0 \leqslant x \leqslant \frac{3}{4}\pi$, and its minimum point M. The shaded region bounded by the curve that lies above the x-axis and the x-axis itself is denoted by R.

(a) Given that the x-coordinate of M lies in the interval $\frac{1}{2}\pi < x < \frac{3}{4}\pi$, find the exact coordinates of M. [4]

(b) Find the exact area of the region R. [4]

23. [9709/w24/32/q11]

Let $f(x) = \dfrac{2e^{2x}}{e^{2x} - 3e^{x} + 2}$.

(a) Find $f'(x)$ and hence find the exact coordinates of the stationary point of the curve with equation $y = f(x)$. [5]

(b) Use the substitution $u = e^{x}$ and partial fractions to find the exact value of $\displaystyle\int_{\ln 3}^{\ln 5} f(x)\,dx$.

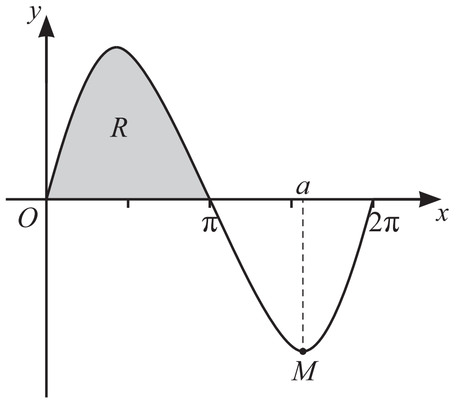
Give your answer in the form $\ln a$, where a is a rational number in its simplest form. [9]

24. [9709/w24/33/q9]

(a) Find the quotient and remainder when $x^4 + 16$ is divided by $x^2 + 4$. [3]

(b) Hence show that $\int_2^{2\sqrt{3}} \frac{x^4 + 16}{x^2 + 4}\,dx = \frac{4}{3}(\pi + 4)$. [5]

25. [9709/w24/33/q11]

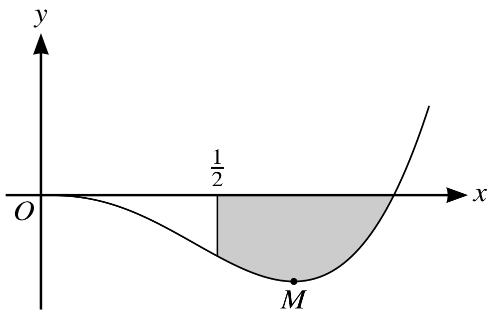


The diagram shows the curve $y = 2\sin x\sqrt{2+\cos x}$, for $0 \leqslant x \leqslant 2\pi$, and its minimum point M, where $x = a$.

(a) Find the value of a correct to 2 decimal places. [5]

(b) Use the substitution $u = 2+\cos x$ to find the exact area of the shaded region R. [6]

26. [9709/m23/32/q8]



The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(a) Find the exact coordinates of M. [4]

(b) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. [5]

27. [9709/m23/32/q11]

Let $f(x) = \dfrac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\int_0^2 f(x)\,dx = \ln 54 - \frac{1}{8}\pi$. [5]

28. [9709/s23/31/q8]

Let $f(x) = \dfrac{3 - 3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are integers. [5]

29. [9709/s23/31/q9]

The constant a is such that $\int_0^a x\mathrm{e}^{-2x}\,\mathrm{d}x = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2}\ln(4a + 2)$. [5]

(b) Verify by calculation that a lies between 0.5 and 1. [2]

(c) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

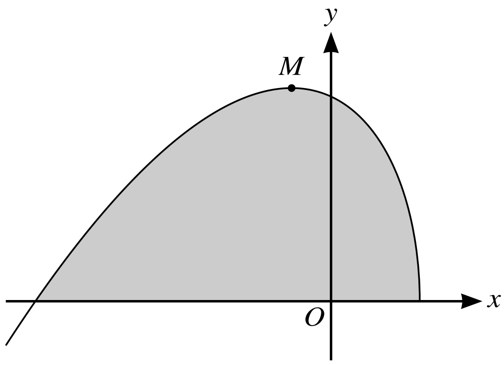
30. [9709/s23/32/q9]

Let $f(x) = \dfrac{2x^2 + 17x - 17}{(1 + 2x)(2 - x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. [5]

31. [9709/s23/32/q10]



The diagram shows the curve $y = (x + 5)\sqrt{3 - 2x}$ and its maximum point M.

(a) Find the exact coordinates of M. [5]

(b) Using the substitution $u = 3 - 2x$, find by integration the area of the shaded region bounded by the curve and the x-axis. Give your answer in the form $a\sqrt{13}$, where a is a rational number. [5]

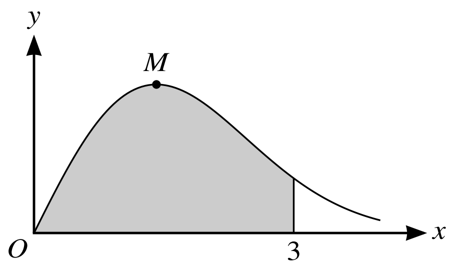
32. [9709/s23/33/q7]

(a) Use the substitution $u = \cos x$ to show that

$$\int_0^{\pi} \sin 2x \, e^{2\cos x} \, dx = \int_{-1}^{1} 2u e^{2u} \, du.$$ [4]

(b) Hence find the exact value of $\int_0^{\pi} \sin 2x \, e^{2\cos x} \, dx$. [4]

33. [9709/w23/31/q9]



The diagram shows the curve $y = xe^{-\frac{1}{4}x^2}$, for $x \geqslant 0$, and its maximum point M.

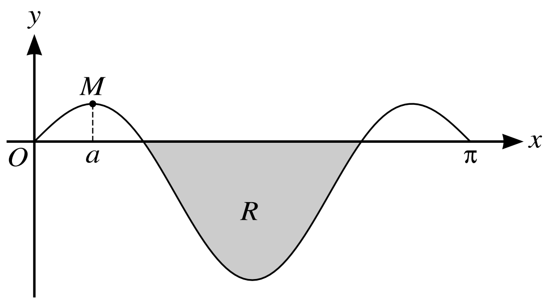
(a) Find the exact coordinates of M. [4]

(b) Using the substitution $x = \sqrt{u}$, or otherwise, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 3$. [5]

34. [9709/w23/32/q5]

Find the exact value of $\displaystyle\int_0^6 \frac{x(x+1)}{x^2+4}\,dx$. [6]

35. [9709/w23/32/q9]

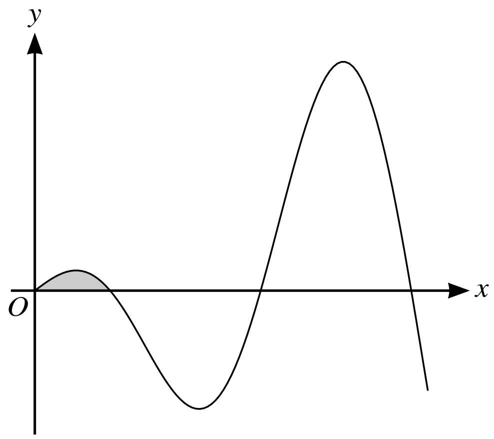


The diagram shows the curve $y = \sin x \cos 2x$, for $0 \leqslant x \leqslant \pi$, and a maximum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Find the value of a correct to 2 decimal places. [5]

(b) Find the exact area of the region R, giving your answer in simplified form. [4]

36. [9709/w23/33/q10]



The diagram shows the curve $y = x\cos 2x$, for $x \geqslant 0$.

(a) Find the equation of the tangent to the curve at the point where $x = \frac{1}{2}\pi$. [4]

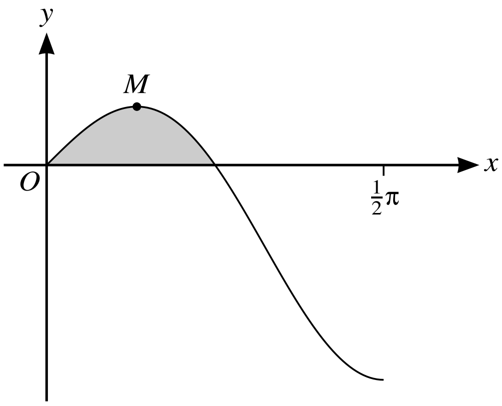
(b) Find the exact area of the shaded region shown in the diagram, bounded by the curve and the x-axis. [5]

37. [9709/m22/32/q8.b]

(a) Find the quotient and remainder when $8x^3 + 4x^2 + 2x + 7$ is divided by $4x^2 + 1$. [3]

(b) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{8x^3 + 4x^2 + 2x + 7}{4x^2 + 1}\,dx$. [5]

38. [9709/m22/32/q11.b]



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 significant figures. [6]

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. [5]

39. [9709/s22/31/q6]

Let $I = \int_0^3 \frac{27}{(9+x^2)^2}\,dx$.

(a) Using the substitution $x = 3\tan\theta$, show that $I = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta$. [4]

(b) Hence find the exact value of I. [4]

40. [9709/s22/32/q8.b]

Let $f(x) = \dfrac{x^2 + 9x}{(3x-1)(x^2+3)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find $\int_1^3 f(x)\,dx$, giving your answer in a simplified exact form. [5]

41. [9709/s22/33/q10.a]

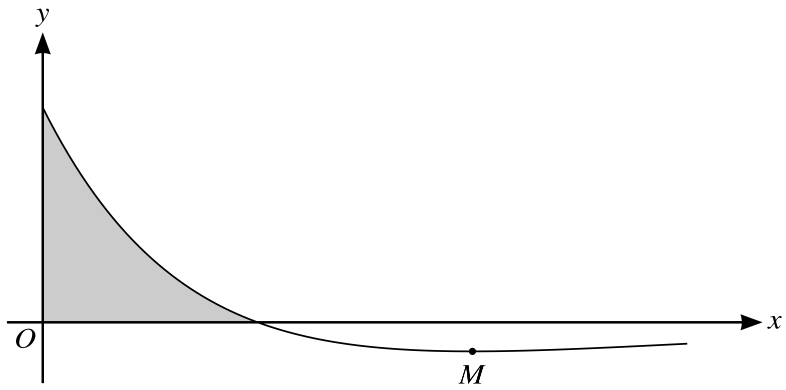
The constant a is such that $\int_1^a x^2 \ln x \, dx = 4$.

(a) Show that $a = \left(\dfrac{35}{3\ln a - 1}\right)^{\frac{1}{3}}$. [5]

(b) Verify by calculation that a lies between 2.4 and 2.8. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

42. [9709/w22/31/q9.b]

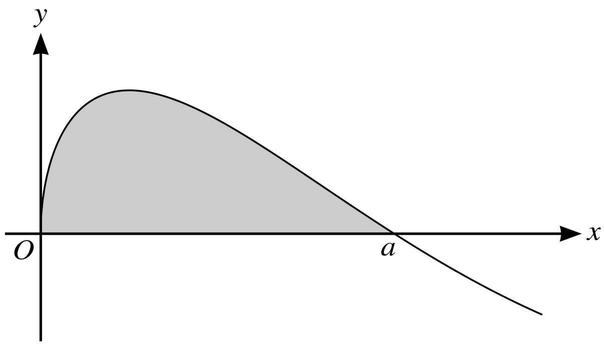


The diagram shows part of the curve $y = (3 - x)e^{-\frac{1}{3}x}$ for $x \geqslant 0$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

(b) Find the area of the shaded region bounded by the curve and the axes, giving your answer in terms of e. [5]

43. [9709/w22/32/q8]



The diagram shows part of the curve $y = \sin\sqrt{x}$. This part of the curve intersects the x-axis at the point where $x = a$.

(a) State the exact value of a. [1]

(b) Using the substitution $u = \sqrt{x}$, find the exact area of the shaded region in the first quadrant bounded by this part of the curve and the x-axis. [7]

44. [9709/w22/32/q10.b]

Let $f(x) = \dfrac{4 - x + x^2}{(1 + x)(2 + x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Find the exact value of $\int_0^4 f(x)\,dx$. Give your answer as a single logarithm. [5]

45. [9709/w22/33/q3]

Find the exact value of $\int_0^{\frac{1}{4}\pi} x \sec^2 x \, dx$. [5]

46. [9709/w22/33/q11.b]

Let $f(x) = \dfrac{5 - x + 6x^2}{(3 - x)(1 + 3x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Find the exact value of $\displaystyle\int_0^1 f(x)\,dx$, simplifying your answer. [5]

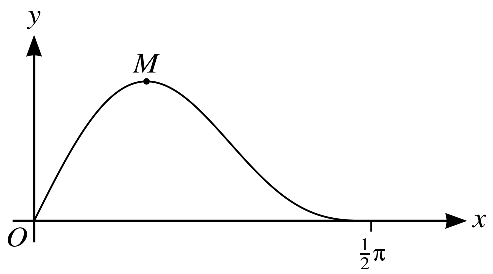
47. [9709/m21/32/q6.b]

Let $f(x) = \dfrac{5a}{(2x-a)(3a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence show that $\displaystyle\int_a^{2a} f(x)\,dx = \ln 6$. [4]

48. [9709/m21/32/q10.a]



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. [5]

(b) Find the exact x-coordinate of M. [6]

49. [9709/s21/31/q4]

(a) Prove that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \tan^2\theta$. [2]

(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\,\mathrm{d}\theta$. [4]

50. [9709/s21/31/q9.b]

The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point. [5]

(b) Show that $\int_1^8 y \, dx = 18 \ln 2 - 9$. [5]

51. [9709/s21/32/q4]

Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right) \mathrm{d}x$. [5]

52. [9709/s21/32/q6.b]

(a) Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$. [3]

(b) Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\,d\theta = \frac{1}{2}\ln 2$. [4]

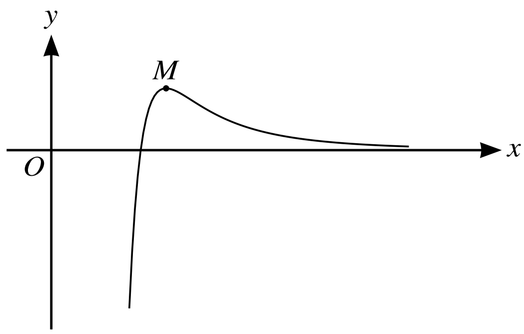
53. [9709/s21/33/q4.b]

Let $f(x) = \dfrac{15 - 6x}{(1 + 2x)(4 - x)}$.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence find $\int_1^2 f(x)\,dx$, giving your answer in the form $\ln\left(\dfrac{a}{b}\right)$, where a and b are integers. [4]

54. [9709/s21/33/q8.b]



The diagram shows the curve $y = \dfrac{\ln x}{x^4}$ and its maximum point M.

(a) Find the exact coordinates of M. [4]

(b) By using integration by parts, show that for all $a > 1$, $\displaystyle\int_1^a \frac{\ln x}{x^4}\,\mathrm{d}x < \frac{1}{9}$. [6]

55. [9709/w21/31/q4]

Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,dx. \qquad [6]$$

56. [9709/w21/32/q6]

(a) Using the expansions of $\sin(3x + 2x)$ and $\sin(3x - 2x)$, show that

$$\tfrac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x.$$ [3]

(b) Hence show that $\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x \, dx = \frac{1}{5}(3 - \sqrt{2})$. [3]

57. [9709/w21/33/q4]

Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, dx$. [5]

58. [9709/w21/33/q9.b]

Let $f(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

(a) Find the x-coordinate of the stationary point of the curve with equation $y = f(x)$. [4]

(b) Using the substitution $u = \sqrt{x}$, show that $\displaystyle\int_0^4 f(x)\,dx = \frac{1}{3}\ln 5$. [6]

59. [9709/m20/32/q4]

Find $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x \sec^2 x \, dx$. Give your answer in a simplified exact form. [7]

60. [9709/s20/31/q5.b]

(a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

(b) Using your answer to part **(a)**, find the exact value of $\displaystyle\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$. [5]

61. [9709/s20/31/q7.b]

Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$. [4]

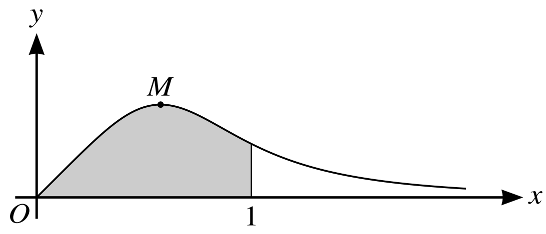
(b) Find $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

62. [9709/s20/32/q3]

Find the exact value of

$$\int_1^4 x^{\frac{3}{2}} \ln x \, dx.$$ [5]

63. [9709/s20/32/q6.b]



The diagram shows the curve $y = \dfrac{x}{1 + 3x^4}$, for $x \geqslant 0$, and its maximum point M.

(a) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [4]

(b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [5]

64. [9709/s20/33/q2]

Find the exact value of $\int_0^1 (2 - x)e^{-2x}\, dx$. [5]

65. [9709/s20/33/q7.c]

Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

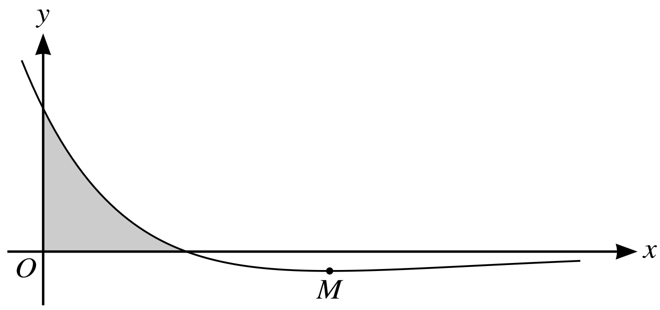
(a) Express $f(x)$ in partial fractions. [2]

(b) Using your answer to part **(a)**, show that

$$\left(f(x)\right)^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

(c) Hence show that $\int_1^2 \left(f(x)\right)^2 \, dx = \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$. [5]

66. [9709/w20/31/q10.b]



The diagram shows the curve $y = (2 - x)e^{-\frac{1}{2}x}$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

(b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e. [5]

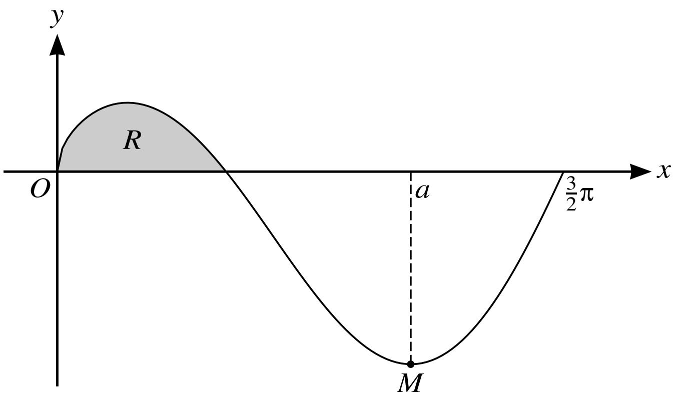
67. [9709/w20/32/q9.b]

Let $f(x) = \dfrac{7x + 18}{(3x + 2)(x^2 + 4)}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\displaystyle\int_0^2 f(x)\,dx$. [6]

68. [9709/w20/32/q10.c]



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \dfrac{1}{2a}$. [3]

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

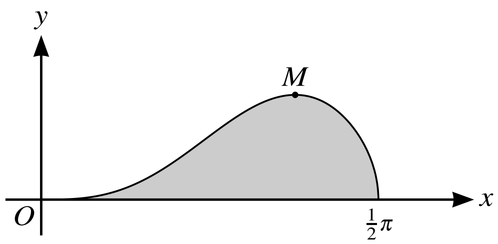
Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

69. [9709/m19/32/q4]

Show that $\int_1^4 x^{-\frac{3}{2}} \ln x \, dx = 2 - \ln 4$. [5]

70. [9709/m19/32/q10.1]



The diagram shows the curve $y = \sin^3 x\sqrt{(\cos x)}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Showing all your working, find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

71. [9709/s19/31/q6]

(i) By first expanding $\sin(2x + x)$, show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$. [4]

(ii) Hence, showing all necessary working, find the exact value of $\int_0^{\frac{1}{3}\pi} \sin^3 x \, dx$. [4]

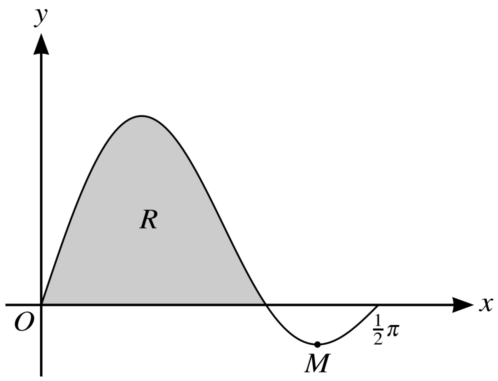
72. [9709/s19/32/q8]

Let $f(x) = \dfrac{10x + 9}{(2x + 1)(2x + 3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^1 f(x)\,dx = \frac{1}{2}\ln\frac{9}{5} + \frac{1}{5}$. [5]

73. [9709/s19/32/q10]



The diagram shows the curve $y = \sin 3x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and its minimum point M. The shaded region R is bounded by the curve and the x-axis.

(i) By expanding $\sin(3x + x)$ and $\sin(3x - x)$ show that

$$\sin 3x \cos x = \tfrac{1}{2}(\sin 4x + \sin 2x).$$ [3]

(ii) Using the result of part **(i)** and showing all necessary working, find the exact area of the region R. [4]

(iii) Using the result of part **(i)**, express $\frac{dy}{dx}$ in terms of $\cos 2x$ and hence find the x-coordinate of M, giving your answer correct to 2 decimal places. [5]

74. [9709/s19/33/q2]

Show that $\int_0^{\frac{1}{4}\pi} x^2 \cos 2x \, dx = \frac{1}{32}(\pi^2 - 8)$. [5]

75. [9709/s19/33/q3]

Let $\mathrm{f}(\theta) = \dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta}$.

(i) Show that $\mathrm{f}(\theta) = \tan\theta$. [3]

(ii) Hence show that $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \mathrm{f}(\theta)\,\mathrm{d}\theta = \frac{1}{2}\ln\frac{3}{2}$. [4]

76. [9709/w19/31/q6]

(i) By differentiating $\frac{\cos x}{\sin x}$, show that if $y = \cot x$ then $\frac{dy}{dx} = -\operatorname{cosec}^2 x$. [2]

(ii) Show that $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} x \operatorname{cosec}^2 x \, dx = \frac{1}{4}(\pi + \ln 4)$. [6]

77. [9709/w19/31/q8]

Let $f(x) = \dfrac{x^2 + x + 6}{x^2(x + 2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing full working, show that the exact value of $\int_1^4 f(x)\,dx$ is $\frac{9}{4}$. [5]

78. [9709/w19/31/q9]

(i) By first expanding $\cos(2x + x)$, show that $\cos 3x \equiv 4\cos^3 x - 3\cos x$. [4]

(ii) Hence solve the equation $\cos 3x + 3\cos x + 1 = 0$, for $0 \leqslant x \leqslant \pi$. [2]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos^3 x \, dx$. [4]

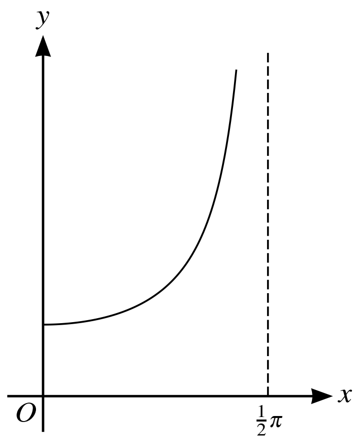
79. [9709/w19/32/q8]

Let $f(x) = \dfrac{2x^2 + x + 8}{(2x-1)(x^2+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing full working, find $\displaystyle\int_1^5 f(x)\,dx$, giving the answer in the form $\ln c$, where c is an integer. [5]

80. [9709/w19/33/q8]



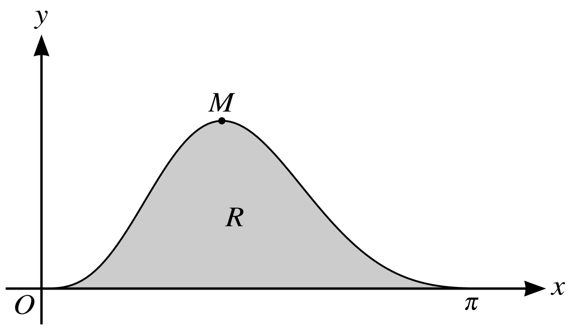
The diagram shows the graph of $y = \sec x$ for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Use the trapezium rule with 2 intervals to estimate the value of $\int_0^{1.2} \sec x \, dx$, giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part **(i)**. [1]

(iii) P is the point on the part of the curve $y = \sec x$ for $0 \leqslant x < \frac{1}{2}\pi$ at which the gradient is 2. By first differentiating $\frac{1}{\cos x}$, find the x-coordinate of P, giving your answer correct to 3 decimal places. [6]

81. [9709/w19/33/q10]



The diagram shows the graph of $y = \mathrm{e}^{\cos x} \sin^3 x$ for $0 \leqslant x \leqslant \pi$, and its maximum point M. The shaded region R is bounded by the curve and the x-axis.

(i) Find the x-coordinate of M. Show all necessary working and give your answer correct to 2 decimal places. [5]

(ii) By first using the substitution $u = \cos x$, find the exact value of the area of R. [7]

82. [9709/m18/32/q1]

Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 - \tan x)}\,\mathrm{d}x,$$

giving your answer correct to 3 decimal places. [3]

83. [9709/m18/32/q3.2]

(i) Using the expansions of $\cos(3x + x)$ and $\cos(3x - x)$, show that

$$\tfrac{1}{2}(\cos 4x + \cos 2x) \equiv \cos 3x \cos x. \qquad [3]$$

(ii) Hence show that $\displaystyle\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x \cos x \, dx = \tfrac{3}{8}\sqrt{3}$. [3]

84. [9709/m18/32/q8.2]

Let $f(x) = \dfrac{5x^2 + x + 27}{(2x+1)(x^2+9)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence find $\int_0^4 f(x)\,dx$, giving your answer in the form $\ln c$, where c is an integer. [5]

85. [9709/s18/31/q5]

Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,\mathrm{d}x$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\,\mathrm{d}\theta$. [4]

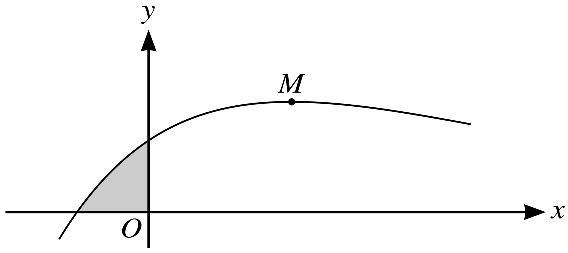
(ii) Hence find the exact value of I. [4]

86. [9709/s18/32/q4]

(i) Show that $\dfrac{2\sin x - \sin 2x}{1 - \cos 2x} \equiv \dfrac{\sin x}{1 + \cos x}$. [4]

(ii) Hence, showing all necessary working, find $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{2\sin x - \sin 2x}{1 - \cos 2x}\,\mathrm{d}x$, giving your answer in the form $\ln k$. [4]

87. [9709/s18/32/q8]



The diagram shows the curve $y = (x+1)e^{-\frac{1}{3}x}$ and its maximum point M.

(i) Find the x-coordinate of M. [4]

(ii) Find the area of the shaded region enclosed by the curve and the axes, giving your answer in terms of e. [5]

88. [9709/s18/33/q3]

Showing all necessary working, find the value of $\int_0^{\frac{1}{6}\pi} x \cos 3x \, dx$, giving your answer in terms of π.

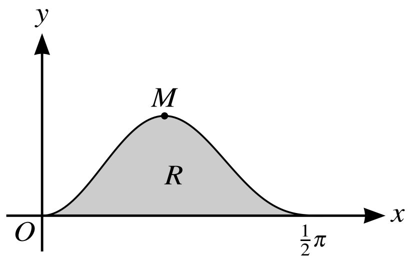
[5]

89. [9709/s18/33/q7]

(i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and $\tan\alpha$. [3]

(ii) Hence, showing all necessary working, show that $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2}\, d\theta = 5.$ [5]

90. [9709/w18/31/q7]



The diagram shows the curve $y = 5\sin^2 x\cos^3 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M. The shaded region R is bounded by the curve and the x-axis.

(i) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [5]

(ii) Using the substitution $u = \sin x$ and showing all necessary working, find the exact area of R. [4]

91. [9709/w18/31/q9]

Let $f(x) = \dfrac{6x^2 + 8x + 9}{(2 - x)(3 + 2x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing all necessary working, show that $\int_{-1}^{0} f(x)\,dx = 1 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$. [5]

92. [9709/w18/32/q3]

(i) Find $\int \frac{\ln x}{x^3}\,dx$. [3]

(ii) Hence show that $\int_1^2 \frac{\ln x}{x^3}\,dx = \frac{1}{16}(3 - \ln 4)$. [2]

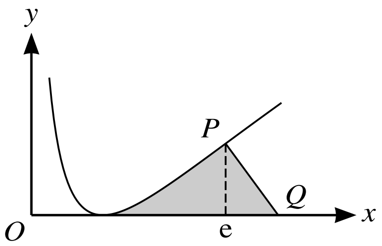
93. [9709/w18/32/q7]

A curve has equation $y = \dfrac{3\cos x}{2 + \sin x}$, for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Find the exact coordinates of the stationary point of the curve. [6]

(ii) The constant a is such that $\displaystyle\int_0^a \frac{3\cos x}{2 + \sin x}\,dx = 1$. Find the value of a, giving your answer correct to 3 significant figures. [4]

94. [9709/m17/32/q10.23]



The diagram shows the curve $y = (\ln x)^2$. The x-coordinate of the point P is equal to e, and the normal to the curve at P meets the x-axis at Q.

(i) Find the x-coordinate of Q. [4]

(ii) Show that $\int \ln x \, dx = x \ln x - x + c$, where c is a constant. [1]

(iii) Using integration by parts, or otherwise, find the exact value of the area of the shaded region between the curve, the x-axis and the normal PQ. [5]

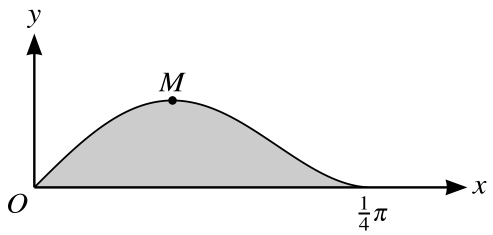
95. [9709/s17/31/q3]

It is given that $x = \ln(1 - y) - \ln y$, where $0 < y < 1$.

(i) Show that $y = \dfrac{e^{-x}}{1 + e^{-x}}$. [2]

(ii) Hence show that $\displaystyle\int_0^1 y \, dx = \ln\left(\frac{2e}{e + 1}\right)$. [4]

96. [9709/s17/31/q10]



The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Find the x-coordinate of M. Give your answer correct to 2 decimal places. [6]

97. [9709/s17/32/q7]

(i) Prove that if $y = \dfrac{1}{\cos\theta}$ then $\dfrac{dy}{d\theta} = \sec\theta\tan\theta$. [2]

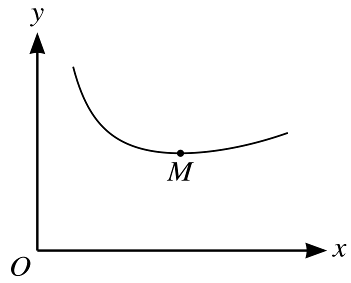
(ii) Prove the identity $\dfrac{1+\sin\theta}{1-\sin\theta} \equiv 2\sec^2\theta + 2\sec\theta\tan\theta - 1$. [3]

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{1+\sin\theta}{1-\sin\theta}\,d\theta$. [4]

98. [9709/s17/33/q4]

Find the exact value of $\int_0^{\frac{1}{2}\pi} \theta \sin \frac{1}{2}\theta \, d\theta$. [4]

99. [9709/s17/33/q7]



The diagram shows a sketch of the curve $y = \dfrac{e^{\frac{1}{2}x}}{x}$ for $x > 0$, and its minimum point M.

(i) Find the x-coordinate of M. [4]

(ii) Use the trapezium rule with two intervals to estimate the value of

$$\int_1^3 \frac{e^{\frac{1}{2}x}}{x}\, dx,$$

giving your answer correct to 2 decimal places. [3]

(iii) The estimate found in part **(ii)** is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with four intervals would be greater than E or less than E. [1]

100. [9709/s17/33/q9]

Let $f(x) = \dfrac{3x^2 - 4}{x^2(3x + 2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_1^2 f(x)\,dx = \ln\left(\frac{25}{8}\right) - 1$. [5]

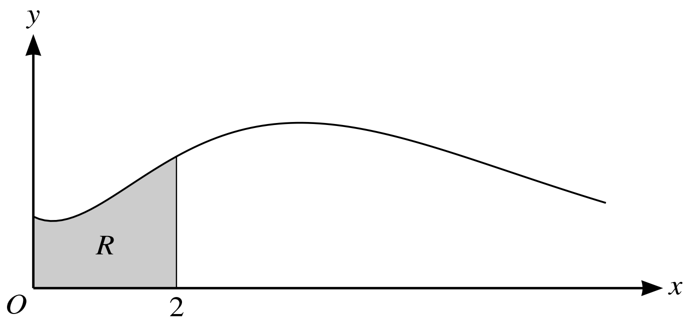
101. [9709/w17/31/q8]

Let $f(x) = \dfrac{4x^2 + 9x - 8}{(x+2)(2x-1)}$.

(i) Express $f(x)$ in the form $A + \dfrac{B}{x+2} + \dfrac{C}{2x-1}$. [4]

(ii) Hence show that $\int_1^4 f(x)\,dx = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$. [5]

102. [9709/w17/31/q9]

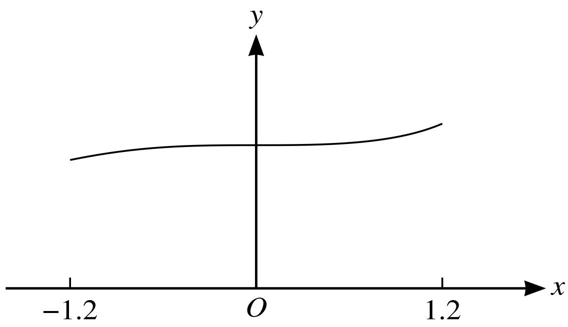


The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(i) Find the exact values of the x-coordinates of the stationary points of the curve. [4]

(ii) Show that the exact value of the area of R is $18 - \dfrac{42}{e}$. [5]

103. [9709/w17/32/q1]



The diagram shows a sketch of the curve $y = \dfrac{3}{\sqrt{(9 - x^3)}}$ for values of x from -1.2 to 1.2.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-1.2}^{1.2} \frac{3}{\sqrt{(9 - x^3)}}\, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

104. [9709/m16/32/q5]

Let $I = \int_0^1 \frac{9}{(3+x^2)^2}\,dx$.

(i) Using the substitution $x = (\sqrt{3})\tan\theta$, show that $I = \sqrt{3}\int_0^{\frac{1}{6}\pi} \cos^2\theta\,d\theta$. [3]

(ii) Hence find the exact value of I. [4]

105. [9709/m16/32/q9.2]

Let $f(x) = \dfrac{3x^3 + 6x - 8}{x(x^2 + 2)}$.

(i) Express $f(x)$ in the form $A + \dfrac{B}{x} + \dfrac{Cx + D}{x^2 + 2}$. [5]

(ii) Show that $\displaystyle\int_1^2 f(x)\,dx = 3 - \ln 4$. [5]

106. [9709/s16/31/q2]

Find the exact value of $\int_0^{\frac{1}{2}} x\mathrm{e}^{-2x}\,\mathrm{d}x$. [5]

107. [9709/s16/32/q3]

Find the exact value of $\int_0^{\frac{1}{2}\pi} x^2 \sin 2x \, dx$. [5]

108. [9709/s16/32/q7]

Let $f(x) = \dfrac{4x^2 + 7x + 4}{(2x+1)(x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\int_0^4 f(x)\,dx = 8 - \ln 3$. [5]

109. [9709/s16/33/q7]

Let $I = \int_0^1 \frac{x^5}{(1+x^2)^3}\, dx$.

(i) Using the substitution $u = 1 + x^2$, show that $I = \int_1^2 \frac{(u-1)^2}{2u^3}\, du$. [3]

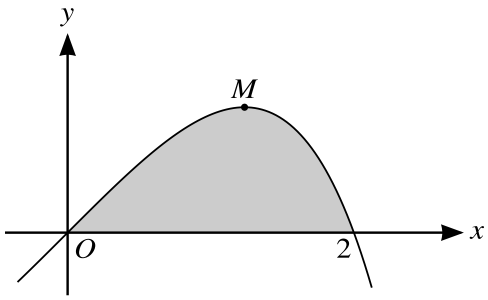
(ii) Hence find the exact value of I. [5]

110. [9709/w16/31/q5]

(i) Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. [4]

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$. [4]

111. [9709/w16/31/q7]



The diagram shows part of the curve $y = (2x - x^2)e^{\frac{1}{2}x}$ and its maximum point M.

(i) Find the exact x-coordinate of M. [4]

(ii) Find the exact value of the area of the shaded region bounded by the curve and the positive x-axis. [5]

112. [9709/w16/33/q6]

Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})}\,dx$.

(i) Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u-1}{u+1}\,du$. [3]

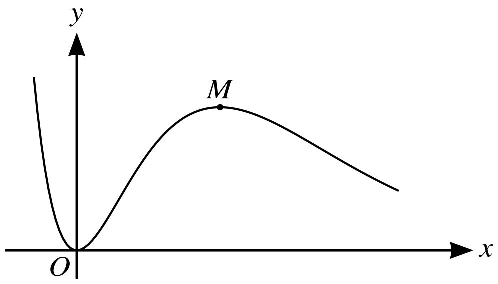
(ii) Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

113. [9709/s15/31/q5]

(a) Find $\int (4 + \tan^2 2x)\,dx$. [3]

(b) Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin\left(x + \frac{1}{6}\pi\right)}{\sin x}\,dx$. [5]

114. [9709/s15/31/q9]



The diagram shows the curve $y = x^2 e^{2-x}$ and its maximum point M.

(i) Show that the x-coordinate of M is 2. [3]

(ii) Find the exact value of $\int_0^2 x^2 e^{2-x} \, dx$. [6]

115. [9709/s15/32/q1]

Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{2}\pi} \ln(1 + \sin x)\, dx,$$

giving your answer correct to 2 decimal places. [3]

116. [9709/s15/32/q6]

Let $I = \int_0^1 \frac{\sqrt{x}}{2 - \sqrt{x}} \, dx$.

(i) Using the substitution $u = 2 - \sqrt{x}$, show that $I = \int_1^2 \frac{2(2-u)^2}{u} \, du$. [4]

(ii) Hence show that $I = 8 \ln 2 - 5$. [4]

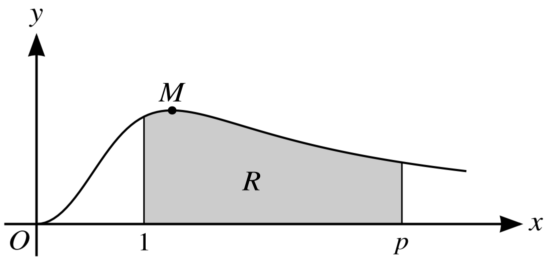
117. [9709/s15/33/q10]

Let $f(x) = \dfrac{11x + 7}{(2x - 1)(x + 2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\int_1^2 f(x)\,dx = \frac{1}{4} + \ln\left(\frac{9}{4}\right)$. [5]

118. [9709/w15/31/q10]



The diagram shows the curve $y = \dfrac{x^2}{1 + x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

119. [9709/w15/33/q5]

Use the substitution $u = 4 - 3\cos x$ to find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4 - 3\cos x)}}\, dx$. [8]

120. [9709/w15/33/q7]

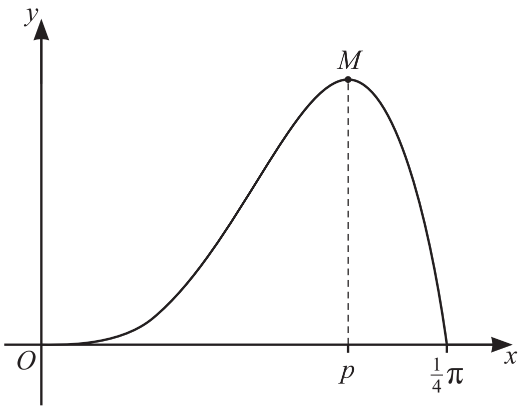
(i) Show that $(x+1)$ is a factor of $4x^3 - x^2 - 11x - 6$. [2]

(ii) Find $\int \frac{4x^2 + 9x - 1}{4x^3 - x^2 - 11x - 6}\,dx$. [8]

Chapter 8

Numerical solution of equations

1. [9709/m25/32/q7]



The diagram shows the curve $y = x^3 \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The curve has a maximum point at M, where $x = p$.

(a) Show that p satisfies the equation $p = \frac{1}{2}\tan^{-1}\left(\frac{3}{2p}\right)$. [3]

(b) Show by calculation that $0.5 < p < 0.7$. [2]

(c) Use an iterative formula based on the equation in part **(a)** to calculate p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

2. [9709/s25/31/q9]

The constant a is such that $\int_1^a 6x \ln x \, dx = 4$.

(a) Show that $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$, where $\exp(x)$ denotes e^x. [5]

(b) Verify by calculation that a lies between 2 and 2.1. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

3. [9709/s25/32/q6]

(a) By sketching a suitable pair of graphs, show that the equation

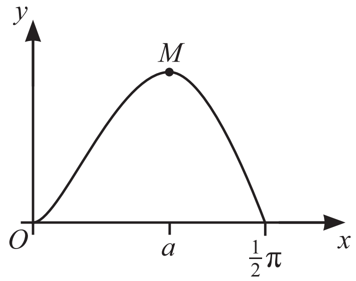
$$|x-2| = 2\sin\tfrac{1}{2}x$$

has only one root in the interval $0 < x < \pi$. [2]

(b) Show by calculation that this root lies between 1 and 1.5. [2]

(c) Use the iterative formula $x_{n+1} = 2 - 2\sin\frac{1}{2}x_n$ with an initial value of 1.03 to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

4. [9709/s25/33/q11]



The diagram shows the curve $y = \sqrt{x}\sin 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curve has a maximum point at M, where $x = a$.

(a) Show that $\tan 2a = -4a$ [4]

(b) Show by calculation that $0\ 9 < a < 0.95$. [2]

(c) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\left(\pi - \tan^{-1}\left(4x_n\right)\right)$$

converges, then it converges to a. [2]

(d) Use the iterative formula in part **(c)** to calculate a correct to 4 decimal places. Give the result of each iteration to 6 decimal places. [3]

5. [9709/s25/35/q8]

(a) By sketching a suitable pair of graphs, show that the equation $\sec 2x = -2x - \frac{1}{2}$ has exactly one root in the interval $0 \leqslant x \leqslant \frac{1}{2}\pi$. [2]

(b) Show by calculation that this root lies between 0.8 and 1.2. [2]

(c) Show that, if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\cos^{-1}\left(\frac{-2}{4x_n + 1}\right)$$

converges, then it converges to the root of the equation in part **(a)**. [2]

(d) Use this iterative formula to calculate this root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

6. [9709/w25/31/q9]

(a) By sketching a suitable pair of graphs, show that the equation

$$\sec 2x = -e^x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(b) Show by calculation that this root lies between 0.9 and 1. [2]

(c) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\cos^{-1}\left(-e^{-x_n}\right)$$

converges, then it converges to the root of the equation in part **(a)**. [1]

(d) Use the iterative formula given in part **(c)** to calculate x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

7. [9709/w25/32/q6]

(a) By sketching a suitable pair of graphs, show that the equation $\cot 2x = 2\sin 2x - 1$ has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(b) Show by calculation that the root is in the interval $0.4 < x < 0.6$. [2]

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(c) Use the iterative formula $x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2\sin 2x_n - 1}\right)$ to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

8. [9709/w25/33/q8]

The curve with equation $y = e^{-5x} \ln 5x$ has a stationary point at $x = p$.

(a) Show that p satisfies the equation $\ln 5p = \dfrac{1}{5p}$. [3]

(b) By sketching a suitable pair of graphs, show that the equation in part **(a)** has only one root. [2]

(c) Show by calculation that $0.2 < p < 0.6$. [2]

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(d) It is given that the equation in part **(a)** can be written in the form $p = \dfrac{1}{5}\exp\left(\dfrac{1}{5p}\right)$, where $\exp(x)$ denotes e^x.

Use an iterative formula based on this rearrangement to calculate p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

9. [9709/w25/35/q10]

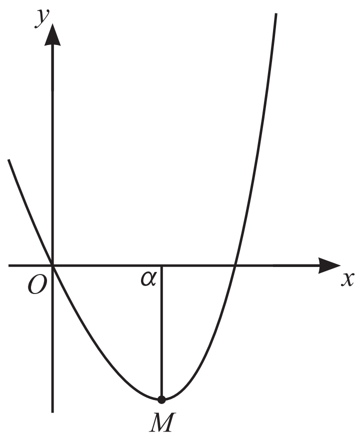
The constant a is such that $\int_0^a x\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x = 6$.

(a) Show that $a = 2 + \mathrm{e}^{-\frac{1}{2}a}$. [5]

(b) Verify by calculation that a lies between 2.2 and 2.4. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

10. [9709/m24/32/q7]



The diagram shows the curve $y = xe^{2x} - 5x$ and its minimum point M, where $x = \alpha$.

(a) Show that α satisfies the equation $\alpha = \frac{1}{2}\ln\left(\frac{5}{1+2\alpha}\right)$. [3]

(b) Verify by calculation that α lies between 0.4 and 0.5 . [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

11. [9709/s24/31/q6]

(a) By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec}\frac{1}{2}x = e^x - 3$ has exactly one root, denoted by α, in the interval $0 < x < \pi$. [2]

(b) Verify by calculation that α lies between 1 and 2. [2]

(c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula

$$x_{n+1} = \ln\left(\operatorname{cosec}\tfrac{1}{2}x_n + 3\right)$$

converges, then it converges to α. [1]

(d) Use this iterative formula with an initial value of 1.4 to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(e) State the minimum number of calculated iterations needed with this initial value to determine α correct to 2 decimal places. [1]

12. [9709/s24/32/q5]

(a) It is given that the equation $e^{2x} = 5 + \cos 3x$ has only one root.

Show by calculation that this root lies in the interval $0.7 < x < 0.8$. [2]

(b) Show that if a sequence of values in the interval $0.7 < x < 0.8$ given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(5 + \cos 3x_n\right)$$

converges then it converges to the root of the equation in part **(a)**. [1]

(c) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

13. [9709/w24/31/q5]

(a) By sketching a suitable pair of graphs, show that the equation $2+e^{-0.2x} = \ln(1+x)$ has only one root. [2]

(b) Show by calculation that this root lies between 7 and 9. [2]

(c) Use the iterative formula

$$x_{n+1} = \exp\left(2+e^{-0.2x_n}\right) - 1$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[exp(x) is an alternative notation for e^x.] [3]

14. [9709/w24/32/q2]

(a) By sketching a suitable pair of graphs, show that the equation $\cot 2x = \sec x$ has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(b) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}(\cos x_n)$$

converges, then it converges to the root in part **(a)**. [1]

15. [9709/w24/33/q2]

Let $f(x) = 2x^3 - 5x^2 + 4$.

(a) Show that if a sequence of values given by the iterative formula

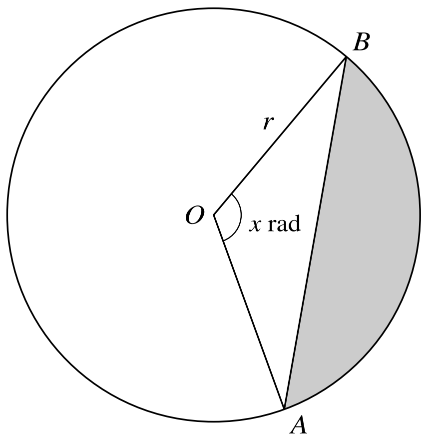
$$x_{n+1} = \sqrt{\frac{4}{5-2x_n}}$$

converges, then it converges to a root of the equation $f(x) = 0$. [2]

(b) The equation has a root close to 1.2 .

Use the iterative formula from part **(a)** and an initial value of 1.2 to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

16. [9709/m23/32/q7]



The diagram shows a circle with centre O and radius r. The angle of the **minor** sector AOB of the circle is x radians. The area of the **major** sector of the circle is 3 times the area of the shaded region.

(a) Show that $x = \frac{3}{4}\sin x + \frac{1}{2}\pi$. [4]

(b) Show by calculation that the root of the equation in **(a)** lies between 2 and 2.5. [2]

(c) Use an iterative formula based on the equation in **(a)** to calculate this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

17. [9709/s23/31/q9]

The constant a is such that $\int_0^a x\mathrm{e}^{-2x}\,\mathrm{d}x = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2}\ln(4a + 2)$. [5]

(b) Verify by calculation that a lies between 0.5 and 1. [2]

(c) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

18. [9709/s23/32/q6]

The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1. [2]

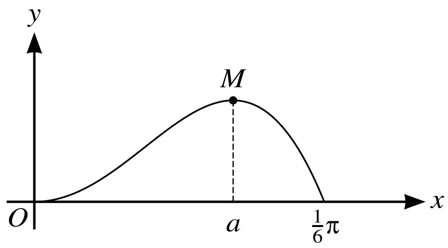
(b) Show that, if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

converges, then it converges to α. [2]

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

19. [9709/s23/33/q5]



The diagram shows the part of the curve $y = x^2 \cos 3x$ for $0 \leqslant x \leqslant \frac{1}{6}\pi$, and its maximum point M, where $x = a$.

(a) Show that a satisfies the equation $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$. [3]

(b) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

20. [9709/w23/31/q8]

(a) By sketching a suitable pair of graphs, show that the equation

$$\sqrt{x} = e^x - 3$$

has only one root. [2]

(b) Show by calculation that this root lies between 1 and 2. [2]

(c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \ln\left(3 + \sqrt{x_n}\right)$$

converges, then it converges to the root of the equation in **(a)**. [1]

(d) Use the iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

21. [9709/w23/32/q6]

(a) By sketching a suitable pair of graphs, show that the equation

$$\cot x = 2 - \cos x$$

has one root in the interval $0 < x \leqslant \frac{1}{2}\pi$. [2]

(b) Show by calculation that this root lies between 0.6 and 0.8. [2]

(c) Use the iterative formula $x_{n+1} = \tan^{-1}\left(\frac{1}{2 - \cos x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

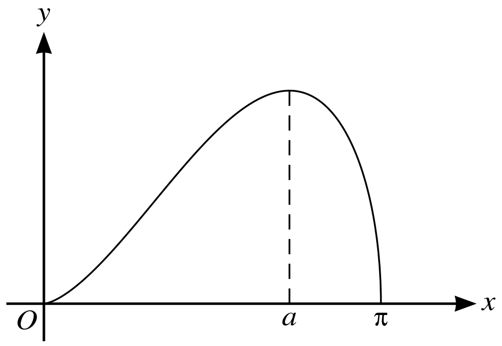
22. [9709/m22/32/q7]

(a) By sketching a suitable pair of graphs, show that the equation $4 - x^2 = \sec \frac{1}{2}x$ has exactly one root in the interval $0 \leqslant x < \pi$. [2]

(b) Verify by calculation that this root lies between 1 and 2. [2]

(c) Use the iterative formula $x_{n+1} = \sqrt{4 - \sec \frac{1}{2}x_n}$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

23. [9709/s22/31/q10]



The curve $y = x\sqrt{\sin x}$ has one stationary point in the interval $0 < x < \pi$, where $x = a$ (see diagram).

(a) Show that $\tan a = -\frac{1}{2}a$. [4]

(b) Verify by calculation that a lies between 2 and 2.5. [2]

(c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi - \tan^{-1}\left(\frac{1}{2}x_n\right)$ converges, then it converges to a, the root of the equation in part **(a)**. [2]

(d) Use the iterative formula given in part **(c)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

24. [9709/s22/32/q5]

(a) By sketching a suitable pair of graphs, show that the equation $\ln x = 3x - x^2$ has one real root. [2]

(b) Verify by calculation that the root lies between 2 and 2.8. [2]

(c) Use the iterative formula $x_{n+1} = \sqrt{3x_n - \ln x_n}$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

25. [9709/s22/33/q10]

The constant a is such that $\int_1^a x^2 \ln x \, dx = 4$.

(a) Show that $a = \left(\frac{35}{3\ln a - 1}\right)^{\frac{1}{3}}$. [5]

(b) Verify by calculation that a lies between 2.4 and 2.8. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

26. [9709/w22/31/q7]

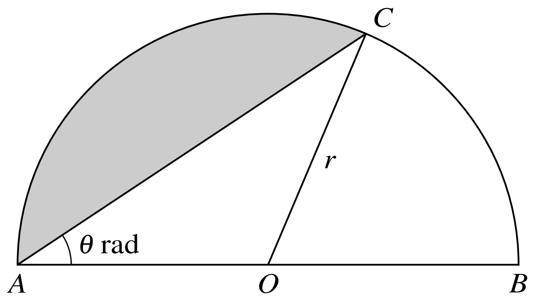
The equation of a curve is $y = \dfrac{x}{\cos^2 x}$, for $0 \leqslant x < \frac{1}{2}\pi$. At the point where $x = a$, the tangent to the curve has gradient equal to 12.

(a) Show that $a = \cos^{-1}\left(\sqrt[3]{\dfrac{\cos a + 2a \sin a}{12}}\right)$. [3]

(b) Verify by calculation that a lies between 0.9 and 1. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

27. [9709/w22/32/q9]



The diagram shows a semicircle with diameter AB, centre O and radius r. The shaded region is the minor segment on the chord AC and its area is one third of the area of the semicircle. The angle CAB is θ radians.

(a) Show that $\theta = \frac{1}{3}(\pi - 1.5 \sin 2\theta)$. [4]

(b) Verify by calculation that $0.5 < \theta < 0.7$. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine θ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

28. [9709/w22/33/q8]

The curve with equation $y = \dfrac{x^3}{e^x - 1}$ has a stationary point at $x = p$, where $p > 0$.

(a) Show that $p = 3(1 - e^{-p})$. [3]

(b) Verify by calculation that p lies between 2.5 and 3. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

29. [9709/m21/32/q9.ab]

Let $f(x) = \dfrac{e^{2x} + 1}{e^{2x} - 1}$, for $x > 0$.

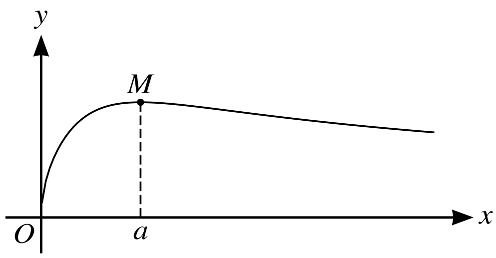
(a) The equation $x = f(x)$ has one root, denoted by a.

Verify by calculation that a lies between 1 and 1.5. [2]

(b) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(c) Find $f'(x)$. Hence find the exact value of x for which $f'(x) = -8$. [6]

30. [9709/s21/31/q7]



The diagram shows the curve $y = \dfrac{\tan^{-1} x}{\sqrt{x}}$ and its maximum point M where $x = a$.

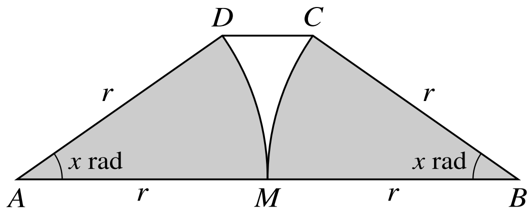
(a) Show that a satisfies the equation

$$a = \tan\left(\frac{2a}{1+a^2}\right).$$ [4]

(b) Verify by calculation that a lies between 1.3 and 1.5. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

31. [9709/s21/32/q10]



The diagram shows a trapezium $ABCD$ in which $AD = BC = r$ and $AB = 2r$. The acute angles BAD and ABC are both equal to x radians. Circular arcs of radius r with centres A and B meet at M, the midpoint of AB.

(a) Given that the sum of the areas of the shaded sectors is 90% of the area of the trapezium, show that x satisfies the equation $x = 0.9(2 - \cos x)\sin x$. [3]

(b) Verify by calculation that x lies between 0.5 and 0.7. [2]

(c) Show that if a sequence of values in the interval $0 < x < \frac{1}{2}\pi$ given by the iterative formula

$$x_{n+1} = \cos^{-1}\left(2 - \frac{x_n}{0.9\sin x_n}\right)$$

converges, then it converges to the root of the equation in part **(a)**. [2]

(d) Use this iterative formula to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

32. [9709/s21/33/q6]

(a) By sketching a suitable pair of graphs, show that the equation $\cot \frac{1}{2}x = 1 + e^{-x}$ has exactly one root in the interval $0 < x \leqslant \pi$. [2]

(b) Verify by calculation that this root lies between 1 and 1.5. [2]

(c) Use the iterative formula $x_{n+1} = 2\tan^{-1}\left(\frac{1}{1 + e^{-x_n}}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

33. [9709/w21/31/q8]

The constant a is such that $\displaystyle\int_1^a \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = 6$.

(a) Show that $a = \exp\left(\dfrac{1}{\sqrt{a}} + 2\right)$. [5]

[$\exp(x)$ is an alternative notation for e^x.]

(b) Verify by calculation that a lies between 9 and 11. [2]

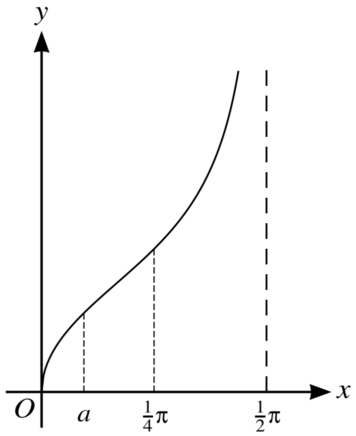
(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

34. [9709/w21/32/q11.c]

The equation of a curve is $y = \sqrt{\tan x}$, for $0 \leqslant x < \frac{1}{2}\pi$.

(a) Express $\frac{dy}{dx}$ in terms of $\tan x$, and verify that $\frac{dy}{dx} = 1$ when $x = \frac{1}{4}\pi$. [4]

The value of $\frac{dy}{dx}$ is also 1 at another point on the curve where $x = a$, as shown in the diagram.



(b) Show that $t^3 + t^2 + 3t - 1 = 0$, where $t = \tan a$. [4]

(c) Use the iterative formula

$$a_{n+1} = \tan^{-1}\left(\frac{1}{3}(1 - \tan^2 a_n - \tan^3 a_n)\right)$$

to determine a correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

35. [9709/w21/33/q10.c]

A large plantation of area $20\,\text{km}^2$ is becoming infected with a plant disease. At time t years the area infected is $x\,\text{km}^2$ and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected.

When $t = 0$, $x = 1$ and $\dfrac{dx}{dt} = 1$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}.$$ [2]

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$. [5]

(c) Use an iterative formula based on the equation in part **(b)**, with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(d) Calculate the value of t at which the entire plantation becomes infected. [1]

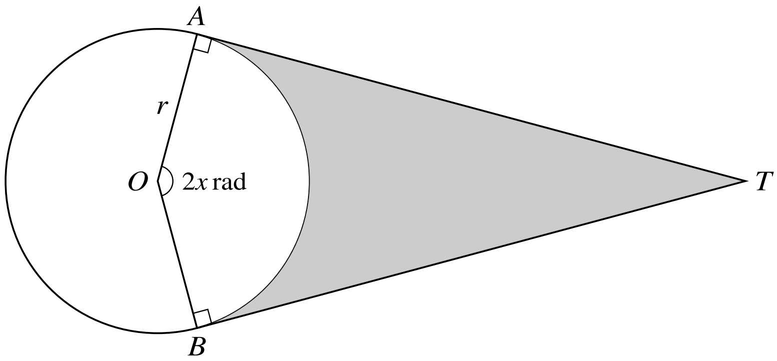
36. [9709/m20/32/q3]

(a) By sketching a suitable pair of graphs, show that the equation $\sec x = 2 - \frac{1}{2}x$ has exactly one root in the interval $0 \leqslant x < \frac{1}{2}\pi$. [2]

(b) Verify by calculation that this root lies between 0.8 and 1. [2]

(c) Use the iterative formula $x_{n+1} = \cos^{-1}\left(\frac{2}{4 - x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

37. [9709/s20/31/q6]



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The area of the shaded region is equal to the area of the circle.

(a) Show that x satisfies the equation $\tan x = \pi + x$. [3]

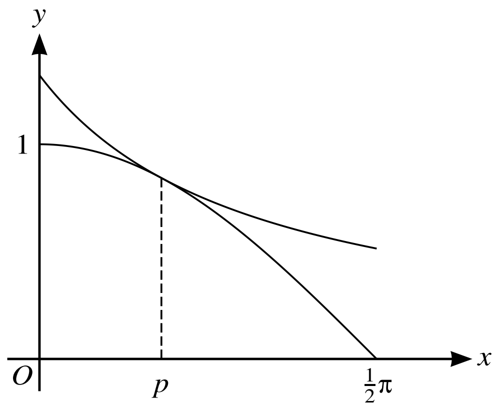
(b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4. [2]

(c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

38. [9709/s20/32/q9]



The diagram shows the curves $y = \cos x$ and $y = \dfrac{k}{1+x}$, where k is a constant, for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = p$.

(a) Show that p satisfies the equation $\tan p = \dfrac{1}{1+p}$. [5]

(b) Use the iterative formula $p_{n+1} = \tan^{-1}\left(\dfrac{1}{1+p_n}\right)$ to determine the value of p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(c) Hence find the value of k correct to 2 decimal places. [2]

39. [9709/s20/33/q6]

(a) By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]

(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part **(a)**. [2]

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

40. [9709/w20/31/q5]

(a) By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$. [2]

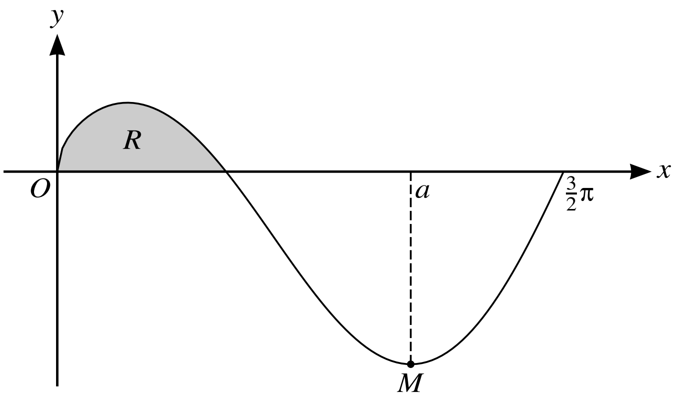
(b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right),$$

with initial value $x_1 = 2$, converges to one of these roots.

Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

41. [9709/w20/32/q10.b]



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \dfrac{1}{2a}$. [3]

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

42. [9709/m19/32/q2]

The sequence of values given by the iterative formula

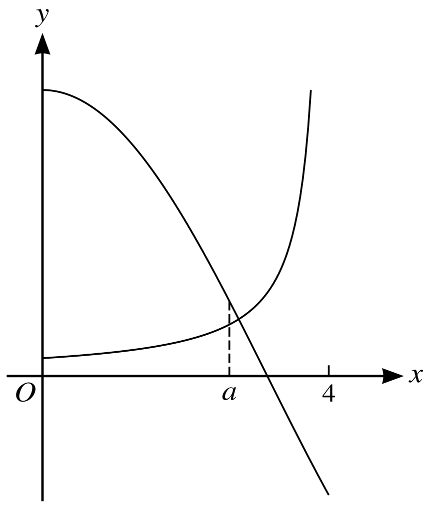
$$x_{n+1} = \frac{2x_n^6 + 12x_n}{3x_n^5 + 8},$$

with initial value $x_1 = 2$, converges to α.

(i) Use the formula to calculate α correct to 4 decimal places. Give the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

43. [9709/s19/31/q7]



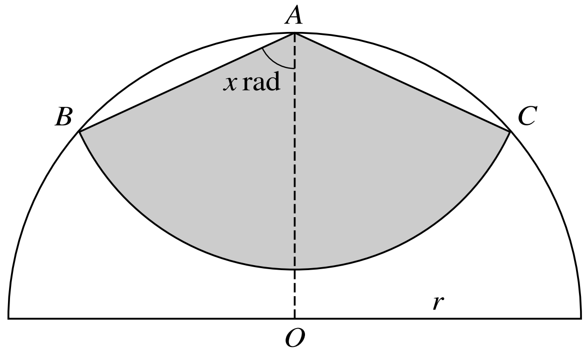
The diagram shows the curves $y = 4\cos\frac{1}{2}x$ and $y = \dfrac{1}{4-x}$, for $0 \leqslant x < 4$. When $x = a$, the tangents to the curves are perpendicular.

(i) Show that $a = 4 - \sqrt{\left(2\sin\frac{1}{2}a\right)}$. [4]

(ii) Verify by calculation that a lies between 2 and 3. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

44. [9709/s19/32/q6]



In the diagram, A is the mid-point of the semicircle with centre O and radius r. A circular arc with centre A meets the semicircle at B and C. The angle OAB is equal to x radians. The area of the shaded region bounded by AB, AC and the arc with centre A is equal to half the area of the semicircle.

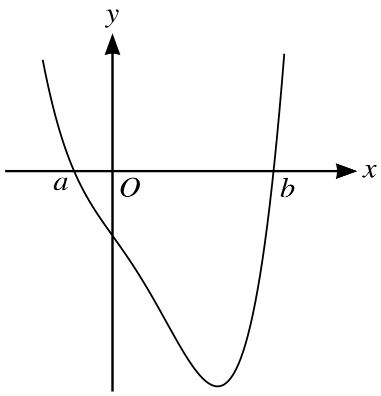
(i) Use triangle OAB to show that $AB = 2r\cos x$. [1]

(ii) Hence show that $x = \cos^{-1}\sqrt{\left(\frac{\pi}{16x}\right)}$. [2]

(iii) Verify by calculation that x lies between 1 and 1.5. [2]

(iv) Use an iterative formula based on the equation in part **(ii)** to determine x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

45. [9709/s19/33/q6]



The diagram shows the curve $y = x^4 - 2x^3 - 7x - 6$. The curve intersects the x-axis at the points $(a, 0)$ and $(b, 0)$, where $a < b$. It is given that b is an integer.

(i) Find the value of b. [1]

(ii) Hence show that a satisfies the equation $a = -\frac{1}{3}(2 + a^2 + a^3)$. [4]

(iii) Use an iterative formula based on the equation in part **(ii)** to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

46. [9709/w19/31/q5]

The curve with equation $y = e^{-2x}\ln(x-1)$ has a stationary point when $x = p$.

(i) Show that p satisfies the equation $x = 1 + \exp\left(\dfrac{1}{2(x-1)}\right)$, where $\exp(x)$ denotes e^x. [3]

(ii) Verify by calculation that p lies between 2.2 and 2.6. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

47. [9709/w19/32/q9]

It is given that $\int_0^a x\cos\frac{1}{3}x\,dx = 3$, where the constant a is such that $0 < a < \frac{3}{2}\pi$.

(i) Show that a satisfies the equation

$$a = \frac{4 - 3\cos\frac{1}{3}a}{\sin\frac{1}{3}a}.$$ [5]

(ii) Verify by calculation that a lies between 2.5 and 3. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to calculate a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

48. [9709/w19/33/q5]

(i) By sketching a suitable pair of graphs, show that the equation $\ln(x+2) = 4e^{-x}$ has exactly one real root. [2]

(ii) Show by calculation that this root lies between $x = 1$ and $x = 1.5$. [2]

(iii) Use the iterative formula $x_{n+1} = \ln\left(\frac{4}{\ln(x_n + 2)}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

49. [9709/m18/32/q7]

(i) By sketching suitable graphs, show that the equation $e^{2x} = 6 + e^{-x}$ has exactly one real root. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1. [2]

(iii) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\ln\left(1 + 6e^{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

50. [9709/s18/31/q8]

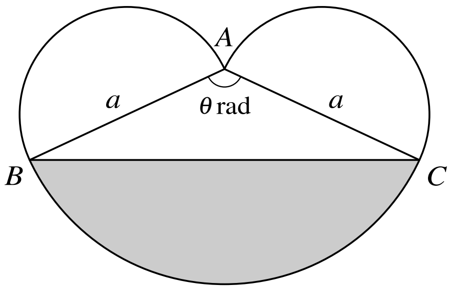
The positive constant a is such that $\int_0^a x\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x = 2$.

(i) Show that a satisfies the equation $a = 2\ln(a + 2)$. [5]

(ii) Verify by calculation that a lies between 3 and 3.5. [2]

(iii) Use an iteration based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

51. [9709/s18/32/q6]



The diagram shows a triangle ABC in which $AB = AC = a$ and angle $BAC = \theta$ radians. Semicircles are drawn outside the triangle with AB and AC as diameters. A circular arc with centre A joins B and C. The area of the shaded segment is equal to the sum of the areas of the semicircles.

(i) Show that $\theta = \frac{1}{2}\pi + \sin\theta$. [3]

(ii) Verify by calculation that θ lies between 2.2 and 2.4. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine θ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

52. [9709/s18/33/q4]

The curve with equation $y = \dfrac{\ln x}{3 + x}$ has a stationary point at $x = p$.

(i) Show that p satisfies the equation $\ln x = 1 + \dfrac{3}{x}$. [3]

(ii) By sketching suitable graphs, show that the equation in part **(i)** has only one root. [2]

(iii) It is given that the equation in part **(i)** can be written in the form $x = \dfrac{3 + x}{\ln x}$. Use an iterative formula based on this rearrangement to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

53. [9709/w18/31/q3]

(i) By sketching a suitable pair of graphs, show that the equation $x^3 = 3 - x$ has exactly one real root. [2]

(ii) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

54. [9709/w18/32/q5]

The equation of a curve is $y = x\ln(8-x)$. The gradient of the curve is equal to 1 at only one point, when $x = a$.

(i) Show that a satisfies the equation $x = 8 - \dfrac{8}{\ln(8-x)}$. [3]

(ii) Verify by calculation that a lies between 2.9 and 3.1. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

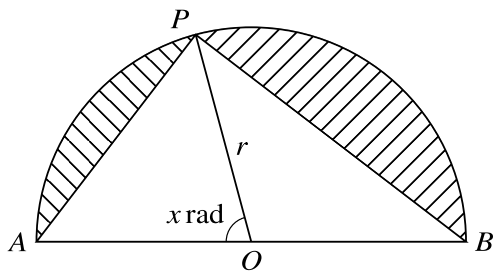
55. [9709/m17/32/q3]

(i) By sketching suitable graphs, show that the equation $e^{-\frac{1}{2}x} = 4 - x^2$ has one positive root and one negative root. [2]

(ii) Verify by calculation that the negative root lies between −1 and −1.5. [2]

(iii) Use the iterative formula $x_{n+1} = -\sqrt{(4 - e^{-\frac{1}{2}x_n})}$ to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

56. [9709/s17/31/q5]



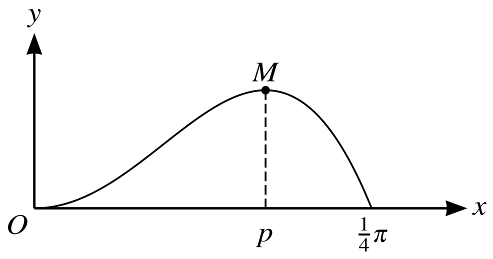
The diagram shows a semicircle with centre O, radius r and diameter AB. The point P on its circumference is such that the area of the minor segment on AP is equal to half the area of the minor segment on BP. The angle AOP is x radians.

(i) Show that x satisfies the equation $x = \frac{1}{3}(\pi + \sin x)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

57. [9709/s17/32/q10]



The diagram shows the curve $y = x^2 \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The curve has a maximum point at M where $x = p$.

(i) Show that p satisfies the equation $p = \frac{1}{2}\tan^{-1}\left(\frac{1}{p}\right)$. [3]

(ii) Use the iterative formula $p_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Find, showing all necessary working, the exact area of the region bounded by the curve and the x-axis. [5]

58. [9709/s17/33/q6]

The equation $\cot x = 1 - x$ has one root in the interval $0 < x < \pi$, denoted by α.

(i) Show by calculation that α is greater than 2.5. [2]

(ii) Show that, if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{1 - x_n}\right)$ converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

59. [9709/w17/31/q3]

The equation $x^3 = 3x + 7$ has one real root, denoted by α.

(i) Show by calculation that α lies between 2 and 3. [2]

Two iterative formulae, A and B, derived from this equation are as follows:

$$x_{n+1} = (3x_n + 7)^{\frac{1}{3}}, \qquad (A)$$

$$x_{n+1} = \frac{x_n^3 - 7}{3}. \qquad (B)$$

Each formula is used with initial value $x_1 = 2.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [4]

60. [9709/w17/32/q9]

It is given that $\int_1^a x^{\frac{1}{2}} \ln x \, dx = 2$, where $a > 1$.

(i) Show that $a^{\frac{3}{2}} = \dfrac{7 + 2a^{\frac{3}{2}}}{3 \ln a}$. [5]

(ii) Show by calculation that a lies between 2 and 4. [2]

(iii) Use the iterative formula

$$a_{n+1} = \left(\frac{7 + 2a_n^{\frac{3}{2}}}{3 \ln a_n} \right)^{\frac{2}{3}}$$

to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

61. [9709/m16/32/q3]

The equation $x^5 - 3x^3 + x^2 - 4 = 0$ has one positive root.

(i) Verify by calculation that this root lies between 1 and 2. [2]

(ii) Show that the equation can be rearranged in the form

$$x = \sqrt[3]{\left(3x + \frac{4}{x^2} - 1\right)}.$$ [1]

(iii) Use an iterative formula based on this rearrangement to determine the positive root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

62. [9709/s16/31/q6]

(i) By sketching a suitable pair of graphs, show that the equation

$$5e^{-x} = \sqrt{x}$$

has one root. [2]

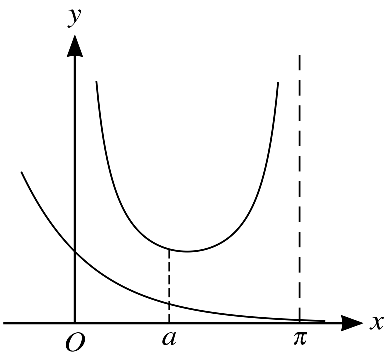
(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(\frac{25}{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

63. [9709/s16/32/q8]



The diagram shows the curve $y = \operatorname{cosec} x$ for $0 < x < \pi$ and part of the curve $y = e^{-x}$. When $x = a$, the tangents to the curves are parallel.

(i) By differentiating $\dfrac{1}{\sin x}$, show that if $y = \operatorname{cosec} x$ then $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$. [3]

(ii) By equating the gradients of the curves at $x = a$, show that

$$a = \tan^{-1}\left(\frac{e^a}{\sin a}\right).$$ [2]

(iii) Verify by calculation that a lies between 1 and 1.5. [2]

(iv) Use an iterative formula based on the equation in part **(ii)** to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

64. [9709/s16/33/q6]

The curve with equation $y = x^2 \cos \frac{1}{2}x$ has a stationary point at $x = p$ in the interval $0 < x < \pi$.

(i) Show that p satisfies the equation $\tan \frac{1}{2}p = \dfrac{4}{p}$. [3]

(ii) Verify by calculation that p lies between 2 and 2.5. [2]

(iii) Use the iterative formula $p_{n+1} = 2\tan^{-1}\left(\dfrac{4}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

65. [9709/w16/31/q6]

(i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} \tfrac{1}{2}x = \tfrac{1}{3}x + 1$$

has one root in the interval $0 < x \leqslant \pi$. [2]

(ii) Show by calculation that this root lies between 1.4 and 1.6. [2]

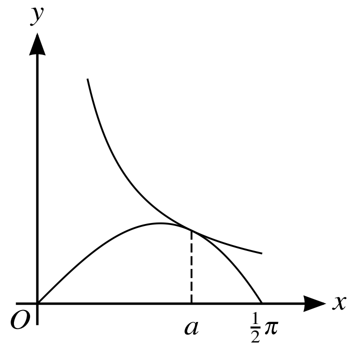
(iii) Show that, if a sequence of values in the interval $0 < x \leqslant \pi$ given by the iterative formula

$$x_{n+1} = 2\sin^{-1}\left(\frac{3}{x_n + 3}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

66. [9709/w16/33/q9]



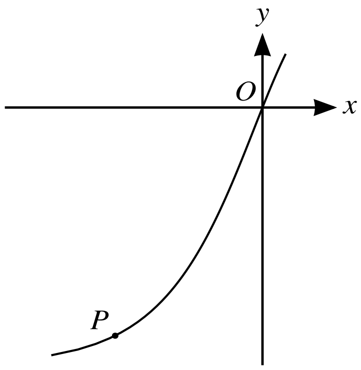
The diagram shows the curves $y = x\cos x$ and $y = \dfrac{k}{x}$, where k is a constant, for $0 < x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = a$.

(i) Show that a satisfies the equation $\tan a = \dfrac{2}{a}$. [5]

(ii) Use the iterative formula $a_{n+1} = \tan^{-1}\left(\dfrac{2}{a_n}\right)$ to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iii) Hence find the value of k correct to 2 decimal places. [2]

67. [9709/s15/31/q10]



The diagram shows part of the curve with parametric equations

$$x = 2\ln(t+2), \qquad y = t^3 + 2t + 3.$$

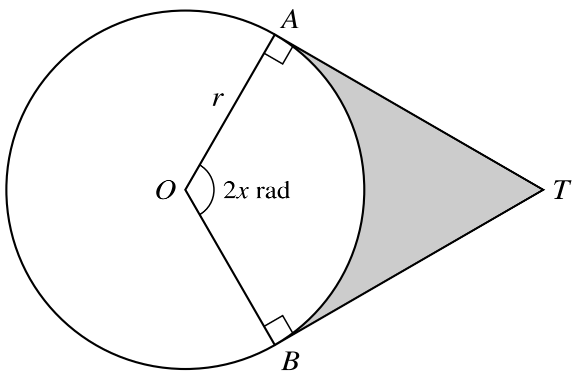
(i) Find the gradient of the curve at the origin. [5]

(ii) At the point P on the curve, the value of the parameter is p. It is given that the gradient of the curve at P is $\frac{1}{2}$.

(a) Show that $p = \dfrac{1}{3p^2+2} - 2$. [1]

(b) By first using an iterative formula based on the equation in part **(a)**, determine the coordinates of the point P. Give the result of each iteration to 5 decimal places and each coordinate of P correct to 2 decimal places. [4]

68. [9709/s15/32/q5]



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and the angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The perimeter of the shaded region is equal to the circumference of the circle.

(i) Show that x satisfies the equation

$$\tan x = \pi - x.$$ [3]

(ii) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.3. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi - x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

69. [9709/s15/33/q6]

It is given that $\int_0^a x\cos x\,dx = 0.5$, where $0 < a < \frac{1}{2}\pi$.

(i) Show that a satisfies the equation $\sin a = \dfrac{1.5 - \cos a}{a}$. [4]

(ii) Verify by calculation that a is greater than 1. [2]

(iii) Use the iterative formula

$$a_{n+1} = \sin^{-1}\left(\frac{1.5 - \cos a_n}{a_n}\right)$$

to determine the value of a correct to 4 decimal places, giving the result of each iteration to 6 decimal places. [3]

70. [9709/w15/31/q4]

The equation $x^3 - x^2 - 6 = 0$ has one real root, denoted by α.

(i) Find by calculation the pair of consecutive integers between which α lies. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

71. [9709/w15/33/q4]

A curve has parametric equations

$$x = t^2 + 3t + 1, \qquad y = t^4 + 1.$$

The point P on the curve has parameter p. It is given that the gradient of the curve at P is 4.

(i) Show that $p = \sqrt[3]{(2p + 3)}$. [3]

(ii) Verify by calculation that the value of p lies between 1.8 and 2.0. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Chapter 9

Vectors

1. [9709/m25/32/q8]

Two lines have equations $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

(b) Find the obtuse angle between the directions of the two lines. [3]

2. [9709/s25/31/q8]

With respect to the origin O, the points A and B have position vectors $2\mathbf{i}+4\mathbf{k}$ and $5\mathbf{i}+\mathbf{j}+6\mathbf{k}$ respectively. The line l_1 passes through the points A and B.

(a) Find a vector equation for the line l_1. [2]

The line l_2 has equation $\mathbf{r} = 2\mathbf{i}+\mathbf{j}+5\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$.

(b) Show that l_1 and l_2 do **not** intersect. [4]

(c) Find the acute angle between the directions of l_1 and l_2. [3]

3. [9709/s25/32/q9]

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

(a) Find a vector equation for the line through A and B. [2]

(b) Using a scalar product, find the exact value of $\cos BAC$. [4]

(c) Hence find the exact area of triangle ABC. [3]

4. [9709/s25/33/q9]

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j}, \quad \overrightarrow{OB} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

The line l passes through B and C.

(a) Find a vector equation for l. [2]

(b) The point P is the foot of the perpendicular from A to l.

Find the position vector of P. [4]

(c) The point D is the reflection of A in l.

Find the position vector of D. [2]

5. [9709/s25/35/q10]

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} - 6\mathbf{k}, \qquad \overrightarrow{OB} = b\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \qquad \text{and} \qquad \overrightarrow{OC} = -4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}.$$

(a) It is given that $|\overrightarrow{AB}| = |\overrightarrow{BC}|$.

Find the value of b. [3]

(b) A, B, C and D are the vertices of a rhombus.

Find the position vector of D. [2]

(c) Calculate angle ABC. [3]

6. [9709/w25/31/q11]

With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \qquad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{OD} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

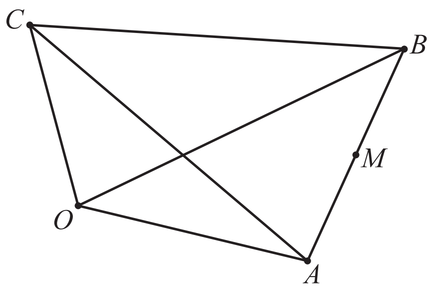
The line m passes through the points A and B.

(a) Find a vector equation for m. [2]

(b) Find the position vector of the point of intersection of m and the line passing through the points C and D. [4]

(c) Find the position vector of the foot of the perpendicular from C to m. [4]

7. [9709/w25/32/q11]



Relative to the origin O, the position vectors of the points A, B and C are

$$\overrightarrow{OA} = 4\mathbf{i} - 2\mathbf{j},\ \overrightarrow{OB} = 2\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}, \text{ and } \overrightarrow{OC} = -2\mathbf{i} + 6\mathbf{k}.$$

The midpoint of AB is M, as shown in the diagram.

(a) Find the vectors $\overrightarrow{MB}$ and $\overrightarrow{MC}$. [2]

(b) Calculate the exact value of the cosine of angle CMB. [3]

(c) Hence or otherwise find the exact area of triangle ABC. [4]

8. [9709/w25/33/q9]

The line l_1 passes through the point (3, 1, −6) and is parallel to the vector $2\mathbf{i}+\mathbf{j}+4\mathbf{k}$.

The line l_2 passes through the point (−1, 3, −6) and is perpendicular to the vector $3\mathbf{i}-2\mathbf{j}+\mathbf{k}$. The direction vector for l_2 has no component in the x-direction.

(a) Write down a vector equation for l_1 and find a vector equation for l_2. [3]

(b) Calculate the acute angle between l_1 and l_2. [3]

(c) Find the position vector of the point of intersection of l_1 and l_2. [3]

9. [9709/w25/35/q9]

The equations of two lines are given by

$$l_1: \ \mathbf{r} = (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}),$$
$$l_2: \ \mathbf{r} = (3\mathbf{i} - \mathbf{j} + 5\mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j} + a\mathbf{k}).$$

(a) Find the value of a for which l_1 is perpendicular to l_2. [2]

(b) Find the value of a for which l_1 and l_2 intersect. [4]

(c) Find the values of a for which the acute angle between l_1 and l_2 is equal to $\cos^{-1}\left(\frac{5}{14}\right)$. [4]

10. [9709/m24/32/q9]

Relative to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 8\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}.$$

(a) Show that $OABC$ is a rectangle. [4]

(b) Use a scalar product to find the acute angle between the diagonals of $OABC$. [4]

11. [9709/s24/31/q9]

The equations of two straight lines l_1 and l_2 are

$$l_1: \quad \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + a\mathbf{k}) \quad \text{and} \quad l_2: \quad \mathbf{r} = -\mathbf{i} - \mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}),$$

where a is a constant.

The lines l_1 and l_2 are perpendicular.

(a) Show that $a = 4$. [1]

The lines l_1 and l_2 also intersect.

(b) Find the position vector of the point of intersection. [4]

The point A has position vector $-5\mathbf{i} + \mathbf{j} - 9\mathbf{k}$.

(c) Show that A lies on l_1. [2]

The point B is the image of A after a reflection in the line l_2.

(d) Find the position vector of B. [2]

12. [9709/s24/32/q8]

The points A, B and C have position vectors $\overrightarrow{OA} = -2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $\overrightarrow{OB} = 5\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OC} = 8\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, where O is the origin. The line l_1 passes through B and C.

(a) Find a vector equation for l_1. [3]

The line l_2 has equation $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

(b) Find the coordinates of the point of intersection of l_1 and l_2. [4]

(c) The point D on l_2 is such that $AB = BD$.

Find the position vector of D. [5]

13. [9709/s24/33/q10]

The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 2a\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} + a\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}),$$

where a is a constant.

(a) Given that the acute angle between the directions of these lines is $\frac{1}{4}\pi$, find the possible values of a. [6]

(b) Given instead that the lines intersect, find the value of a and the position vector of the point of intersection. [5]

14. [9709/w24/31/q9]

The position vector of point A relative to the origin O is $\overrightarrow{OA} = 8\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$.
The line l passes through A and is parallel to the vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

(a) State a vector equation for l. [2]

(b) The position vector of point B relative to the origin O is $\overrightarrow{OB} = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$, where t is a constant. The line l also passes through B.

Find the value of t. [3]

(c) The line m has vector equation $\mathbf{r} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(a\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. The acute angle between the directions of l and m is θ, where $\cos\theta = \frac{1}{\sqrt{6}}$.

Find the possible values of a. [5]

15. [9709/w24/32/q9]

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}.$$

(a) The point D is such that $ABCD$ is a trapezium with $\overrightarrow{DC} = 3\overrightarrow{AB}$.

Find the position vector of D. [2]

(b) The diagonals of the trapezium intersect at the point P.

Find the position vector of P. [5]

(c) Using a scalar product, calculate angle ABC. [4]

16. [9709/w24/33/q6]

The lines l and m have vector equations

$$l: \quad \mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{k}) \quad \text{and} \quad m: \quad \mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + 5\mathbf{k}).$$

Lines l and m intersect at the point P.

(a) State the coordinates of P. [1]

(b) Find the exact value of the cosine of the acute angle between l and m. [3]

(c) The point A on line l has coordinates $(0, 1, 1)$. The point B on line m has coordinates $(0, 2, -8)$.

Find the exact area of triangle APB. [3]

17. [9709/m23/32/q10]

With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix}.$$

(a) Find the obtuse angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [3]

The line l passes through the points A and B.

(b) Find a vector equation for the line l. [2]

(c) Find the position vector of the point of intersection of the line l and the line passing through C and D. [4]

18. [9709/s23/31/q6]

Relative to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

The quadrilateral $ABCD$ is a parallelogram.

(a) Find the position vector of D. [3]

(b) The angle between BA and BC is θ.

Find the exact value of $\cos\theta$. [3]

(c) Hence find the area of $ABCD$, giving your answer in the form $p\sqrt{q}$, where p and q are integers. [4]

19. [9709/s23/32/q11]

The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$.

(a) Show that l does not intersect the line passing through A and B. [5]

(b) Find the position vector of the foot of the perpendicular from A to l. [4]

20. [9709/s23/33/q9]

The lines l and m have equations

$$l: \quad \mathbf{r} = a\mathbf{i} + 3\mathbf{j} + b\mathbf{k} + \lambda(c\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}),$$
$$m: \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

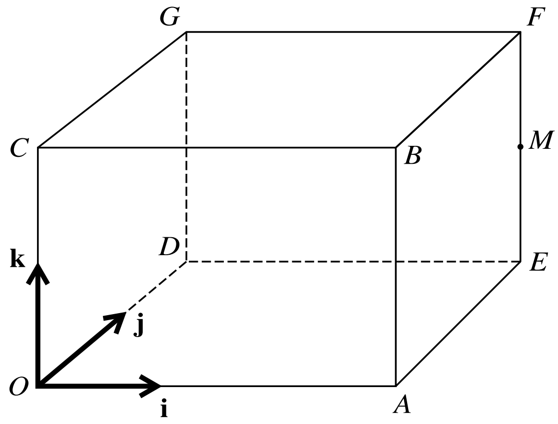
Relative to the origin O, the position vector of the point P is $4\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$.

(a) Given that l is perpendicular to m and that P lies on l, find the values of the constants a, b and c. [4]

(b) The perpendicular from P meets line m at Q. The point R lies on PQ extended, with $PQ : QR = 2 : 3$.

Find the position vector of R. [6]

21. [9709/w23/31/q11]



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 3$ units, $OC = 2$ units and $OD = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OD and OC respectively. M is the midpoint of EF.

(a) Find the position vector of M. [1]

The position vector of P is $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

(b) Calculate angle PAM. [4]

(c) Find the exact length of the perpendicular from P to the line passing through O and M. [5]

22. [9709/w23/32/q10]

The equations of the lines l and m are given by

$$l: \ \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad m: \ \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix},$$

where c is a positive constant. It is given that the angle between l and m is 60°.

(a) Find the value of c. [4]

(b) Show that the length of the perpendicular from (6, −3, 6) to l is $\sqrt{11}$. [5]

23. [9709/w23/33/q11]

The line l has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The points A and B have position vectors $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively.

(a) Find a unit vector in the direction of l. [2]

The line m passes through the points A and B.

(b) Find a vector equation for m. [2]

(c) Determine whether lines l and m are parallel, intersect or are skew. [5]

24. [9709/m22/32/q10]

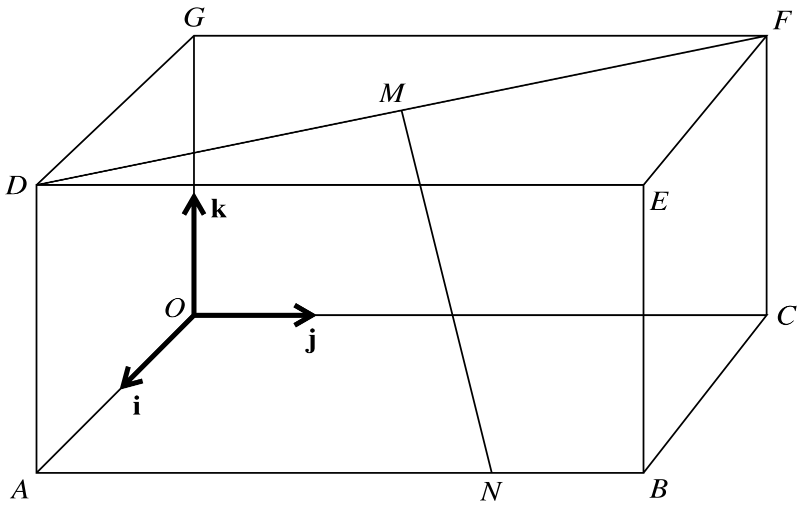
The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ respectively. The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$.

(a) Find a vector equation for the line through A and B. [3]

(b) Find the acute angle between the directions of AB and l, giving your answer in degrees. [3]

(c) Show that the line through A and B does not intersect the line l. [4]

25. [9709/s22/31/q9]



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 4$ units and $OG = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OC and OG respectively. The point M is the midpoint of DF. The point N on AB is such that $AN = 3NB$.

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**. [3]

(b) Find a vector equation for the line through M and N. [2]

(c) Show that the length of the perpendicular from O to the line through M and N is $\sqrt{\frac{53}{6}}$. [4]

26. [9709/s22/32/q9]

The lines l and m have vector equations

$$\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

respectively, where a and b are constants.

(a) Given that l and m intersect, show that $2b - a = 4$. [4]

(b) Given also that l and m are perpendicular, find the values of a and b. [4]

(c) When a and b have these values, find the position vector of the point of intersection of l and m. [2]

27. [9709/s22/33/q9]

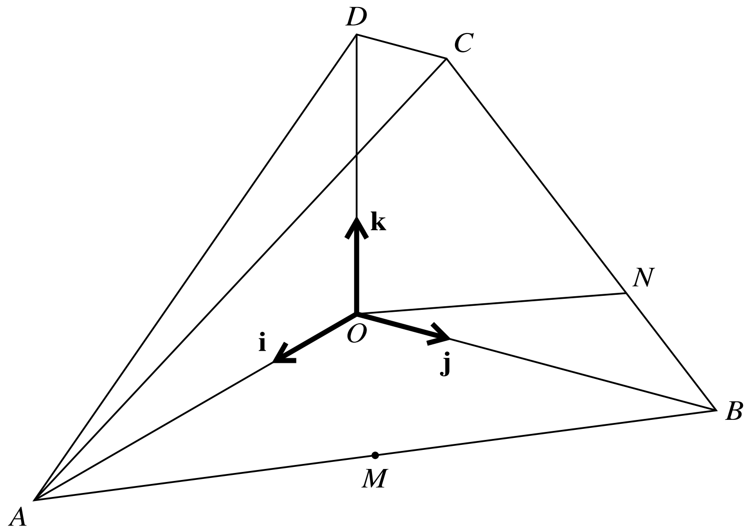
With respect to the origin O, the point A has position vector given by $\overrightarrow{OA} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. The line l has vector equation $\mathbf{r} = 4\mathbf{i} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(a) Find in degrees the acute angle between the directions of OA and l. [3]

(b) Find the position vector of the foot of the perpendicular from A to l. [4]

(c) Hence find the position vector of the reflection of A in l. [2]

28. [9709/w22/31/q11]



In the diagram, $OABCD$ is a solid figure in which $OA = OB = 4$ units and $OD = 3$ units. The edge OD is vertical, DC is parallel to OB and $DC = 1$ unit. The base, OAB, is horizontal and angle $AOB = 90°$. Unit vectors **i**, **j** and **k** are parallel to OA, OB and OD respectively. The midpoint of AB is M and the point N on BC is such that $CN = 2NB$.

(a) Express vectors $\overrightarrow{MD}$ and $\overrightarrow{ON}$ in terms of **i**, **j** and **k**. [4]

(b) Calculate the angle in degrees between the directions of $\overrightarrow{MD}$ and $\overrightarrow{ON}$. [3]

(c) Show that the length of the perpendicular from M to ON is $\sqrt{\frac{22}{5}}$. [4]

29. [9709/w22/32/q6]

Relative to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}.$$

(a) Using a scalar product, find the cosine of angle BAC. [4]

(b) Hence find the area of triangle ABC. Give your answer in a simplified exact form. [4]

30. [9709/w22/33/q9]

With respect to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}.$$

The midpoint of AC is M and the point N lies on BC, between B and C, and is such that $BN = 2NC$.

(a) Find the position vectors of M and N. [3]

(b) Find a vector equation for the line through M and N. [2]

(c) Find the position vector of the point Q where the line through M and N intersects the line through A and B. [4]

31. [9709/m21/32/q7]

Two lines have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

(b) Find the acute angle between the directions of the two lines. [3]

32. [9709/s21/31/q8]

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(a) Find the acute angle between the directions of AB and l. [4]

(b) Find the position vector of the point P on l such that $AP = BP$. [5]

33. [9709/s21/32/q11]

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OB} = \mathbf{j} - 2\mathbf{k}$.

(a) Show that $OA = OB$ and use a scalar product to calculate angle AOB in degrees. [4]

The midpoint of AB is M. The point P on the line through O and M is such that $PA : OA = \sqrt{7} : 1$.

(b) Find the possible position vectors of P. [6]

34. [9709/s21/33/q9]

The quadrilateral $ABCD$ is a trapezium in which AB and DC are parallel. With respect to the origin O, the position vectors of A, B and C are given by $\overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(a) Given that $\overrightarrow{DC} = 3\overrightarrow{AB}$, find the position vector of D. [3]

(b) State a vector equation for the line through A and B. [1]

(c) Find the distance between the parallel sides and hence find the area of the trapezium. [5]

35. [9709/w21/31/q9]

Two lines l and m have equations $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + s(4\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + t(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ respectively.

(a) Show that l and m are perpendicular. [2]

(b) Show that l and m intersect and state the position vector of the point of intersection. [5]

(c) Show that the length of the perpendicular from the origin to the line m is $\frac{1}{3}\sqrt{5}$. [4]

36. [9709/w21/32/q10]

With respect to the origin O, the position vectors of the points A and B are given by $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

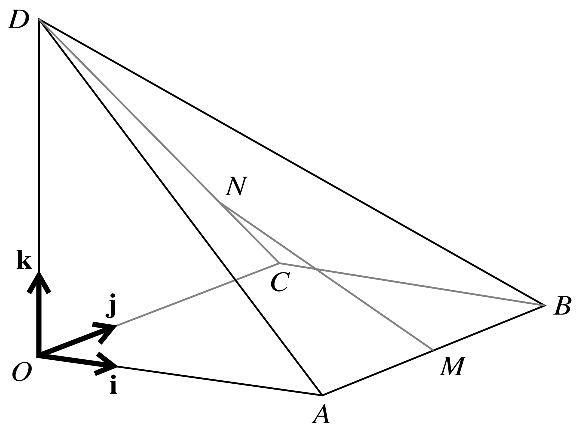
(a) Find a vector equation for the line l through A and B. [3]

(b) The point C lies on l and is such that $\overrightarrow{AC} = 3\overrightarrow{AB}$.

Find the position vector of C. [2]

(c) Find the possible position vectors of the point P on l such that $OP = \sqrt{14}$. [5]

37. [9709/w21/33/q8]



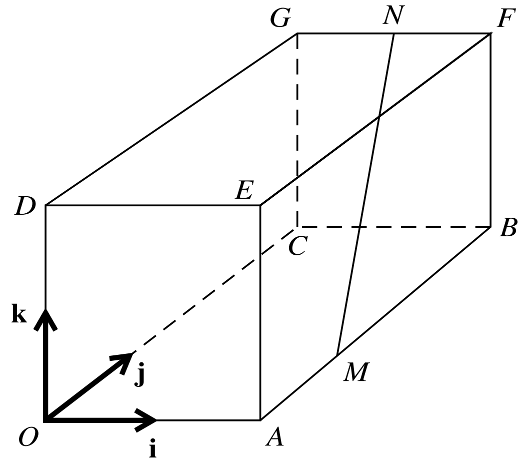
In the diagram, $OABCD$ is a pyramid with vertex D. The horizontal base $OABC$ is a square of side 4 units. The edge OD is vertical and $OD = 4$ units. The unit vectors **i**, **j** and **k** are parallel to OA, OC and OD respectively.

The midpoint of AB is M and the point N on CD is such that $DN = 3NC$.

(a) Find a vector equation for the line through M and N. [5]

(b) Show that the length of the perpendicular from O to MN is $\frac{1}{3}\sqrt{82}$. [4]

38. [9709/m20/32/q8]



In the diagram, *OABCDEFG* is a cuboid in which $OA = 2$ units, $OC = 3$ units and $OD = 2$ units. Unit vectors **i**, **j** and **k** are parallel to *OA*, *OC* and *OD* respectively. The point *M* on *AB* is such that $MB = 2AM$. The midpoint of *FG* is *N*.

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**. [3]

(b) Find a vector equation for the line through *M* and *N*. [2]

(c) Find the position vector of *P*, the foot of the perpendicular from *D* to the line through *M* and *N*. [4]

39. [9709/s20/31/q9]

With respect to the origin O, the vertices of a triangle ABC have position vectors

$$\overrightarrow{OA} = 2\mathbf{i} + 5\mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

(a) Using a scalar product, show that angle ABC is a right angle. [3]

(b) Show that triangle ABC is isosceles. [2]

(c) Find the exact length of the perpendicular from O to the line through B and C. [4]

40. [9709/s20/32/q10]

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. The midpoint of OA is M. The point N lying on AB, between A and B, is such that $AN = 2NB$.

(a) Find a vector equation for the line through M and N. [5]

The line through M and N intersects the line through O and B at the point P.

(b) Find the position vector of P. [3]

(c) Calculate angle OPM, giving your answer in degrees. [3]

41. [9709/s20/33/q8]

Relative to the origin O, the points A, B and D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{k}.$$

A fourth point C is such that $ABCD$ is a parallelogram.

(a) Find the position vector of C and verify that the parallelogram is not a rhombus. [5]

(b) Find angle BAD, giving your answer in degrees. [3]

(c) Find the area of the parallelogram correct to 3 significant figures. [2]

42. [9709/w20/31/q11]

Two lines have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$, where a is a constant.

(a) Given that the two lines intersect, find the value of a and the position vector of the point of intersection. [5]

(b) Given instead that the acute angle between the directions of the two lines is $\cos^{-1}\left(\frac{1}{6}\right)$, find the two possible values of a. [6]

43. [9709/w20/32/q8]

With respect to the origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Show that $AB = 2CD$. [3]

(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3]

(c) Show that the line through A and B does not intersect the line through C and D. [4]

44. [9709/s19/32/q9.1]

The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

(ii) The plane m is perpendicular to AB and passes through the mid-point of AB. The plane m intersects the line l at the point P. Find the equation of m and the position vector of P. [5]

45. [9709/s19/33/q10.1]

The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$.

(i) The point P has position vector $4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Find the length of the perpendicular from P to l. [5]

(ii) It is given that l lies in the plane with equation $ax + by + 2z = 13$, where a and b are constants. Find the values of a and b. [6]

46. [9709/w19/31/q7.1]

Two lines l and m have equations $\mathbf{r} = a\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ respectively, where a is a constant. It is given that the lines intersect.

(i) Find the value of a. [4]

(ii) When a has this value, find the equation of the plane containing l and m. [5]

47. [9709/s18/31/q10.1]

The point P has position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(i) Find the length of the perpendicular from P to l, giving your answer correct to 3 significant figures. [5]

(ii) Find the equation of the plane containing l and P, giving your answer in the form $ax + by + cz = d$. [5]

48. [9709/s18/32/q10.1]

Two lines l and m have equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + s(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ respectively.

(i) Show that the lines are skew. [4]

A plane p is parallel to the lines l and m.

(ii) Find a vector that is normal to p. [3]

(iii) Given that p is equidistant from the lines l and m, find the equation of p. Give your answer in the form $ax + by + cz = d$. [3]

49. [9709/s18/33/q10]

The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

The point P, with parameter t, lies on l and is such that angle PAB is equal to $120°$.

(ii) Show that $3t^2 + 8t + 4 = 0$. Hence find the position vector of P. [6]

50. [9709/s17/32/q9.1]

Relative to the origin O, the point A has position vector given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The line l has equation $\mathbf{r} = 9\mathbf{i} - \mathbf{j} + 8\mathbf{k} + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

(i) Find the position vector of the foot of the perpendicular from A to l. Hence find the position vector of the reflection of A in l. [5]

(ii) Find the equation of the plane through the origin which contains l. Give your answer in the form $ax + by + cz = d$. [3]

(iii) Find the exact value of the perpendicular distance of A from this plane. [3]

51. [9709/s17/33/q10.1]

The points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$, where m is a constant.

(i) Given that the line l intersects the line passing through A and B, find the value of m. [5]

(ii) Find the equation of the plane which is parallel to $\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and contains the points A and B. Give your answer in the form $ax + by + cz = d$. [5]

52. [9709/w17/31/q10.12]

The equations of two lines l and m are $\mathbf{r} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ respectively.

(i) Show that the lines do not intersect. [3]

(ii) Calculate the acute angle between the directions of the lines. [3]

(iii) Find the equation of the plane which passes through the point $(3, -2, -1)$ and which is parallel to both l and m. Give your answer in the form $ax + by + cz = d$. [5]

53. [9709/s16/32/q9.1]

The points A, B and C have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$. A fourth point D is such that the quadrilateral $ABCD$ is a parallelogram.

(i) Find the position vector of D and verify that the parallelogram is a rhombus. [5]

(ii) The plane p is parallel to OA and the line BC lies in p. Find the equation of p, giving your answer in the form $ax + by + cz = d$. [5]

54. [9709/s16/33/q8]

The points A and B have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{k}$. The line l has vector equation $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Show that the line passing through A and B does not intersect l. [4]

(ii) Show that the length of the perpendicular from A to l is $\dfrac{1}{\sqrt{2}}$. [5]

55. [9709/w16/33/q10.1]

The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

(i) Find the position vectors of the two points on the line whose distance from the origin is $\sqrt{(10)}$. [5]

(ii) The plane p has equation $ax + y + z = 5$, where a is a constant. The acute angle between the line l and the plane p is equal to $\sin^{-1}\left(\frac{2}{3}\right)$. Find the possible values of a. [5]

56. [9709/s15/31/q6]

The straight line l_1 passes through the points $(0,\ 1,\ 5)$ and $(2,\ -2,\ 1)$. The straight line l_2 has equation $\mathbf{r} = 7\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$.

(i) Show that the lines l_1 and l_2 are skew. [6]

(ii) Find the acute angle between the direction of the line l_2 and the direction of the x-axis. [3]

57. [9709/s15/32/q10.1]

The points A and B have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

(ii) Find the equation of the plane containing the line l and the point A. Give your answer in the form $ax + by + cz = d$. [6]

58. [9709/w15/31/q7.1]

The points A, B and C have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

The plane m is perpendicular to AB and contains the point C.

(i) Find a vector equation for the line passing through A and B. [2]

(ii) Obtain the equation of the plane m, giving your answer in the form $ax + by + cz = d$. [2]

(iii) The line through A and B intersects the plane m at the point N. Find the position vector of N and show that $CN = \sqrt{(13)}$. [5]

Chapter 10

Differential equations

1. [9709/m25/32/q6]

The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = \left(\frac{1}{5}x + 1\right)\sin^2 2\theta,$$

and $x = 5$ when $\theta = 0$.

Solve the differential equation and obtain an expression for x in terms of θ. [7]

2. [9709/s25/31/q10]

(a) Find the quotient and remainder when $x^3 + 5x^2 - 2x - 15$ is divided by $x^2 - 3$. [3]

(b) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{x^3 + 5x^2 - 2x - 15}{6y(x^2 - 3)}.$$

It is given that $y = 2$ when $x = 2$.

Solve the differential equation to obtain an expression for y^2 in terms of x. [5]

3. [9709/s25/32/q8]

The variables x and θ satisfy the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (4x+3)\cos 2\theta,$$

and $x = 0$ when $\theta = \frac{1}{12}\pi$.

Solve the differential equation and obtain an expression for x in terms of θ. [7]

4. [9709/s25/33/q10]

The variables x and y satisfy the differential equation

$$\sin 4y \frac{dy}{dx} = x \sin 2y \sin 3x.$$

It is given that $y = \frac{1}{12}\pi$ when $x = \frac{1}{2}\pi$.

(a) Solve the differential equation, obtaining a relation between x and y. [8]

(b) Given that $0 < y < \frac{1}{2}\pi$, find the values of y when $x = 0$. [2]

5. [9709/s25/35/q11]

The variables x and y satisfy the differential equation

$$(x^2+3)\frac{dy}{dx} = e^{3y}(x-2).$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation, and find the value of y when $x = 2$. [8]

6. [9709/w25/31/q8]

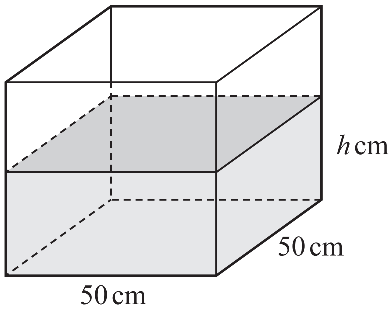
The variables x and y satisfy the differential equation

$$\left(x^2+1\right)\frac{dy}{dx} = kxe^{2y},$$

where k is a constant. It is given that $y = 0$ when $x = 0$ and that $y = -\frac{1}{2}$ when $x = 1$.

Solve the differential equation and find the exact value of y when $x = \sqrt{3}$. [8]

7. [9709/w25/32/q10]



The diagram shows a tank for holding water. The tank is in the shape of a cube of side 50 cm. At time t seconds, the depth of water in the tank is h cm. Water is poured into the tank at a rate of $5000\,\text{cm}^3\,\text{s}^{-1}$. Water pours out of the tank through a hole in the bottom at a rate proportional to h^2.

When $h = 20$, the depth of the water is increasing at a rate of $0.4\,\text{cm}\,\text{s}^{-1}$.

(a) Show that $\dfrac{dh}{dt} = \dfrac{500 - h^2}{250}$. [4]

(b) Given that $h = 0$ when $t = 0$, find the time taken for the depth of the water in the tank to reach 20 cm. [5]

8. [9709/w25/33/q10]

(a) Express $\frac{2}{1-9y^2}$ in partial fractions. [2]

(b) The variables x and y satisfy the differential equation

$$2\cos^2 3x \frac{dy}{dx} = 1 - 9y^2,$$

and $y = 0$ when $x = \frac{1}{12}\pi$.

Solve the differential equation and obtain an expression for y in terms of x. [6]

9. [9709/w25/35/q11]

A fungal disease is affecting some of the trees in a forest. The fraction of the trees affected after t years is denoted by x. The rate of increase of x is proportional to the product of the fraction of the trees affected and the fraction of the trees **not** affected.

(a) Explain why, after t years, $\frac{dx}{dt} = kx(1-x)$, where k is a constant. [1]

(b) When the disease is first detected, one quarter of the trees are affected.
Two years later, one third of the trees are affected.

Solve the differential equation to find the number of years from the time when the disease is first detected until the time when three quarters of the trees are affected. Give your answer correct to the nearest year. [8]

10. [9709/m24/32/q11]

The variables y and θ satisfy the differential equation

$$(1+y)(1+\cos 2\theta)\frac{dy}{d\theta} = e^{3y}.$$

It is given that $y = 0$ when $\theta = \frac{1}{4}\pi$.

Solve the differential equation and find the exact value of $\tan\theta$ when $y = 1$. [9]

11. [9709/s24/31/q11]

In a field there are 300 plants of a certain species, all of which can be infected by a particular disease. At time t after the first plant is infected there are x infected plants. The rate of change of x is proportional to the product of the number of plants infected and the number of plants that are **not** yet infected. The variables x and t are treated as continuous, and it is given that $\frac{dx}{dt} = 0.2$ and $x = 1$ when $t = 0$.

(a) Show that x and t satisfy the differential equation

$$1495\frac{dx}{dt} = x(300 - x).$$ [2]

(b) Using partial fractions, solve the differential equation and obtain an expression for t in terms of a single logarithm involving x. [9]

12. [9709/s24/32/q10]

(a) By writing $y = \sec^3\theta$ as $\dfrac{1}{\cos^3\theta}$, show that $\dfrac{dy}{d\theta} = 3\sin\theta\sec^4\theta$. [2]

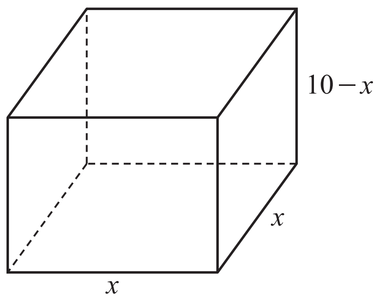
(b) The variables x and θ satisfy the differential equation

$$\left(x^2+9\right)\sin\theta\frac{d\theta}{dx} = (x+3)\cos^4\theta.$$

It is given that $x = 3$ when $\theta = \frac{1}{3}\pi$.

Solve the differential equation to find the value of $\cos\theta$ when $x = 0$. Give your answer correct to 3 significant figures. [8]

13. [9709/s24/33/q9]



A container in the shape of a cuboid has a square base of side x and a height of $(10-x)$. It is given that x varies with time, t, where $t > 0$. The container decreases in volume at a rate which is inversely proportional to t.

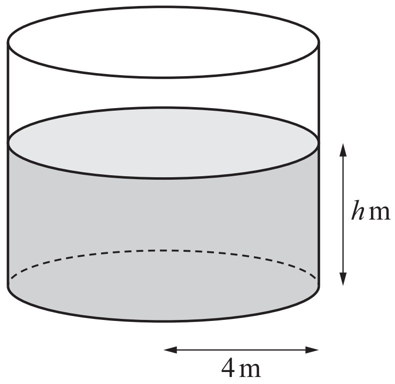
When $t = \frac{1}{10}$, $x = \frac{1}{2}$ and the rate of decrease of x is $\frac{20}{37}$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{-1}{2t(20x - 3x^2)}.$$ [5]

(b) Solve the differential equation, obtaining an expression for t in terms of x. [6]

14. [9709/w24/31/q10]



A large cylindrical tank is used to store water. The base of the tank is a circle of radius 4 metres. At time t minutes, the depth of the water in the tank is h metres. There is a tap at the bottom of the tank. When the tap is open, water flows out of the tank at a rate proportional to the square root of the volume of water in the tank.

(a) Show that $\frac{dh}{dt} = -\lambda\sqrt{h}$, where λ is a positive constant. [4]

(b) At time $t = 0$ the tap is opened. It is given that $h = 4$ when $t = 0$ and that $h = 2.25$ when $t = 20$.

Solve the differential equation to obtain an expression for t in terms of h, and hence find the time taken to empty the tank. [6]

15. [9709/w24/32/q10]

A balloon in the shape of a sphere has volume V and radius r. Air is pumped into the balloon at a constant rate of 40π starting when time $t = 0$ and $r = 0$. At the same time, air begins to flow out of the balloon at a rate of $0.8\pi r$. The balloon remains a sphere at all times.

(a) Show that r and t satisfy the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{50-r}{5r^2}.$$ [3]

(b) Find the quotient and remainder when $5r^2$ is divided by $50-r$. [3]

(c) Solve the differential equation in part **(a)**, obtaining an expression for t in terms of r. [6]

(d) Find the value of t when the radius of the balloon is 12. [1]

16. [9709/w24/33/q10]

A water tank is in the shape of a cuboid with base area $40\,000\,\text{cm}^2$. At time t minutes the depth of water in the tank is h cm. Water is pumped into the tank at a rate of $50\,000\,\text{cm}^3$ per minute. Water is leaking out of the tank through a hole in the bottom at a rate of $600h\,\text{cm}^3$ per minute.

(a) Show that $200\dfrac{dh}{dt} = 250 - 3h$. [3]

(b) It is given that when $t = 0$, $h = 50$.

Find the time taken for the depth of water in the tank to reach 80 cm. Give your answer correct to 2 significant figures. [5]

17. [9709/m23/32/q9]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{3y}\sin^2 2x.$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation and find the value of y when $x = \frac{1}{2}$. [7]

18. [9709/s23/31/q7]

The variables x and y satisfy the differential equation

$$\cos 2x \frac{dy}{dx} = \frac{4 \tan 2x}{\sin^2 3y},$$

where $0 \leqslant x < \frac{1}{4}\pi$. It is given that $y = 0$ when $x = \frac{1}{6}\pi$.

Solve the differential equation to obtain the value of x when $y = \frac{1}{6}\pi$. Give your answer correct to 3 decimal places. [8]

19. [9709/s23/32/q8]

(a) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{4 + 9y^2}{e^{2x+1}}.$$

It is given that $y = 0$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) State what happens to the value of y as x tends to infinity. Give your answer in an exact form. [1]

20. [9709/s23/33/q8]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y^2 + 4}{x(y + 4)}$$

for $x > 0$. It is given that $x = 4$ when $y = 2\sqrt{3}$.

Solve the differential equation to obtain the value of x when $y = 2$. [8]

21. [9709/w23/31/q7]

The variables x and θ satisfy the differential equation

$$\frac{x}{\tan\theta}\frac{dx}{d\theta} = x^2 + 3.$$

It is given that $x = 1$ when $\theta = 0$.

Solve the differential equation, obtaining an expression for x^2 in terms of θ. [7]

22. [9709/w23/32/q11]

The variables x and y satisfy the differential equation

$$x^2\frac{dy}{dx} + y^2 + y = 0.$$

It is given that $x = 1$ when $y = 1$.

(a) Solve the differential equation to obtain an expression for y in terms of x. [8]

(b) State what happens to the value of y when x tends to infinity. Give your answer in an exact form. [1]

23. [9709/w23/33/q8]

The variables x and y satisfy the differential equation

$$e^{4x}\frac{dy}{dx} = \cos^2 3y.$$

It is given that $y = 0$ when $x = 2$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

24. [9709/m22/32/q9]

The variables x and y satisfy the differential equation

$$(x+1)(3x+1)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$.

Solve the differential equation and find the exact value of y when $x = 3$, giving your answer in a simplified form. [9]

25. [9709/s22/31/q4]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1 + x^2},$$

and $y = 2$ when $x = 0$.

Solve the differential equation, obtaining a simplified expression for y in terms of x. [7]

26. [9709/s22/32/q6]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = xe^{y-x},$$

and $y = 0$ when $x = 0$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) Find the value of y when $x = 1$, giving your answer in the form $a - \ln b$, where a and b are integers. [1]

27. [9709/s22/33/q8]

At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by a differential equation of the form $\frac{dN}{dt} = kN^{\frac{3}{2}} \cos 0.02t$, where k is a constant and N is a continuous variable. It is given that when $t = 0$, $N = 100$.

(a) Solve the differential equation, obtaining a relation between N, k and t. [5]

(b) Given also that $N = 625$ when $t = 50$, find the value of k. [2]

(c) Obtain an expression for N in terms of t, and find the greatest value of N predicted by this model. [2]

28. [9709/w22/31/q8]

In a certain chemical reaction the amount, x grams, of a substance is increasing. The differential equation satisfied by x and t, the time in seconds since the reaction began, is

$$\frac{dx}{dt} = kxe^{-0.1t},$$

where k is a positive constant. It is given that $x = 20$ at the start of the reaction.

(a) Solve the differential equation, obtaining a relation between x, t and k. [5]

(b) Given that $x = 40$ when $t = 10$, find the value of k and find the value approached by x as t becomes large. [3]

29. [9709/w22/32/q7]

The variables x and θ satisfy the differential equation

$$x\sin^2\theta\frac{\mathrm{d}x}{\mathrm{d}\theta} = \tan^2\theta - 2\cot\theta,$$

for $0 < \theta < \frac{1}{2}\pi$ and $x > 0$. It is given that $x = 2$ when $\theta = \frac{1}{4}\pi$.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cot^2\theta) = -\frac{2\cot\theta}{\sin^2\theta}$.

(You may assume without proof that the derivative of $\cot\theta$ with respect to θ is $-\operatorname{cosec}^2\theta$.) [1]

(b) Solve the differential equation and find the value of x when $\theta = \frac{1}{6}\pi$. [7]

30. [9709/w22/33/q10]

A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

The differential equation satisfied by V and t is of the form $\frac{\mathrm{d}V}{\mathrm{d}t} = a - bV$.

(a) Write down the values of the constants a and b. [1]

(b) Solve the differential equation and find the value of t when $V = 1000$. [6]

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large. [2]

31. [9709/m21/32/q4]

The variables x and y satisfy the differential equation

$$(1 - \cos x)\frac{\mathrm{d}y}{\mathrm{d}x} = y \sin x.$$

It is given that $y = 4$ when $x = \pi$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [6]

(b) Sketch the graph of y against x for $0 < x < 2\pi$. [1]

32. [9709/s21/31/q10]

The variables x and t satisfy the differential equation $\frac{dx}{dt} = x^2(1 + 2x)$, and $x = 1$ when $t = 0$.

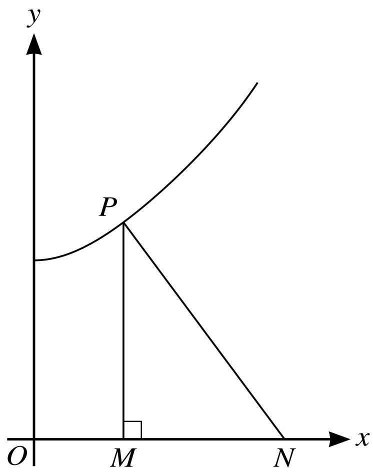
Using partial fractions, solve the differential equation, obtaining an expression for t in terms of x. [11]

33. [9709/s21/32/q7]

A curve is such that the gradient at a general point with coordinates (x, y) is proportional to $\frac{y}{\sqrt{x+1}}$. The curve passes through the points with coordinates $(0, 1)$ and $(3, \mathrm{e})$.

By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [7]

34. [9709/s21/33/q7]



For the curve shown in the diagram, the normal to the curve at the point P with coordinates (x, y) meets the x-axis at N. The point M is the foot of the perpendicular from P to the x-axis.

The curve is such that for all values of x in the interval $0 \leqslant x < \frac{1}{2}\pi$, the area of triangle PMN is equal to $\tan x$.

(a) **(i)** Show that $\dfrac{MN}{y} = \dfrac{dy}{dx}$. [1]

(ii) Hence show that x and y satisfy the differential equation $\frac{1}{2}y^2\dfrac{dy}{dx} = \tan x$. [2]

(b) Given that $y = 1$ when $x = 0$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

35. [9709/w21/31/q7]

(a) Given that $y = \ln(\ln x)$, show that

$$\frac{dy}{dx} = \frac{1}{x \ln x}.$$ [1]

The variables x and t satisfy the differential equation

$$x \ln x + t\frac{dx}{dt} = 0.$$

It is given that $x = e$ when $t = 2$.

(b) Solve the differential equation obtaining an expression for x in terms of t, simplifying your answer. [7]

(c) Hence state what happens to the value of x as t tends to infinity. [1]

36. [9709/w21/32/q7]

The variables x and y satisfy the differential equation

$$e^{2x}\frac{dy}{dx} = 4xy^2,$$

and it is given that $y = 1$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

37. [9709/w21/33/q10]

A large plantation of area 20 km^2 is becoming infected with a plant disease. At time t years the area infected is x km^2 and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected.

When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}.$$ [2]

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$. [5]

(c) Use an iterative formula based on the equation in part **(b)**, with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(d) Calculate the value of t at which the entire plantation becomes infected. [1]

38. [9709/m20/32/q6]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{1 + 4y^2}{e^x}.$$

It is given that $y = 0$ when $x = 1$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) State what happens to the value of y as x tends to infinity. [1]

39. [9709/s20/31/q8]

A certain curve is such that its gradient at a point (x, y) is proportional to $\frac{y}{x\sqrt{x}}$. The curve passes through the points with coordinates $(1, 1)$ and $(4, \mathrm{e})$.

(a) By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [8]

(b) Describe what happens to y as x tends to infinity. [1]

40. [9709/s20/32/q7]

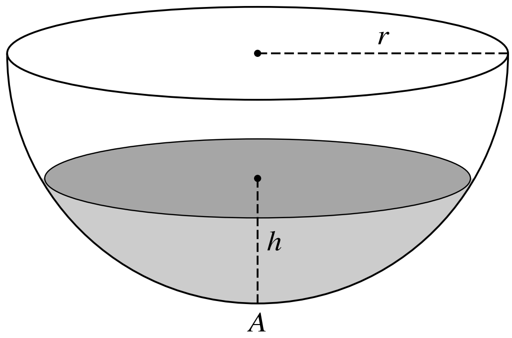
The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y-1}{(x+1)(x+3)}.$$

It is given that $y = 2$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [9]

41. [9709/s20/33/q10]



A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r, as shown in the diagram. The depth of water at time t is h. At time $t = 0$ the tank is full and the depth of the water is r. At this instant a tap at A is opened and water begins to flow out at a rate proportional to $\sqrt{h}$. The tank becomes empty at time $t = 14$.

The volume of water in the tank is V when the depth is h. It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

(a) Show that h and t satisfy a differential equation of the form

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where B is a positive constant. [4]

(b) Solve the differential equation and obtain an expression for t in terms of h and r. [8]

42. [9709/w20/31/q8]

The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x\frac{dy}{dx} = (1 - 2x^2)y,$$

for $x > 0$. It is given that $y = 1$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [6]

43. [9709/w20/32/q7]

The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

(b) State what happens to the value of x when t tends to infinity. [1]

44. [9709/m19/32/q6]

The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky^3\mathrm{e}^{-x},$$

where k is a constant. It is given that $y = 1$ when $x = 0$, and that $y = \sqrt{\mathrm{e}}$ when $x = 1$. Solve the differential equation, obtaining an expression for y in terms of x. [7]

45. [9709/s19/31/q5]

(i) Differentiate $\dfrac{1}{\sin^2\theta}$ with respect to θ. [2]

(ii) The variables x and θ satisfy the differential equation

$$x\tan\theta\frac{dx}{d\theta} + \operatorname{cosec}^2\theta = 0,$$

for $0 < \theta < \frac{1}{2}\pi$ and $x > 0$. It is given that $x = 4$ when $\theta = \frac{1}{6}\pi$. Solve the differential equation, obtaining an expression for x in terms of θ. [6]

46. [9709/s19/32/q7]

The variables x and y satisfy the differential equation $\frac{dy}{dx} = xe^{x+y}$. It is given that $y = 0$ when $x = 0$.

(i) Solve the differential equation, obtaining y in terms of x. [7]

(ii) Explain why x can only take values that are less than 1. [1]

47. [9709/s19/33/q5]

The variables x and y satisfy the differential equation

$$(x+1)y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 5.$$

It is given that $y = 2$ when $x = 0$. Solve the differential equation obtaining an expression for y^2 in terms of x. [7]

48. [9709/w19/31/q4]

The number of insects in a population t weeks after the start of observations is denoted by N. The population is decreasing at a rate proportional to $Ne^{-0.02t}$. The variables N and t are treated as continuous, and it is given that when $t = 0$, $N = 1000$ and $\frac{dN}{dt} = -10$.

(i) Show that N and t satisfy the differential equation

$$\frac{dN}{dt} = -0.01e^{-0.02t}N.$$ [1]

(ii) Solve the differential equation and find the value of t when $N = 800$. [6]

(iii) State what happens to the value of N as t becomes large. [1]

49. [9709/w19/32/q6]

The variables x and θ satisfy the differential equation

$$\sin\tfrac{1}{2}\theta\,\frac{\mathrm{d}x}{\mathrm{d}\theta} = (x+2)\cos\tfrac{1}{2}\theta$$

for $0 < \theta < \pi$. It is given that $x = 1$ when $\theta = \frac{1}{3}\pi$. Solve the differential equation and obtain an expression for x in terms of $\cos\theta$. [8]

50. [9709/w19/33/q9]

The variables x and t satisfy the differential equation $5\frac{\mathrm{d}x}{\mathrm{d}t} = (20 - x)(40 - x)$. It is given that $x = 10$ when $t = 0$.

(i) Using partial fractions, solve the differential equation, obtaining an expression for x in terms of t. [9]

(ii) State what happens to the value of x when t becomes large. [1]

51. [9709/m18/32/q6]

The variables x and θ satisfy the differential equation

$$x\cos^2\theta\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\tan\theta + 1,$$

for $0 \leqslant \theta < \frac{1}{2}\pi$ and $x > 0$. It is given that $x = 1$ when $\theta = \frac{1}{4}\pi$.

(i) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\tan^2\theta) = \frac{2\tan\theta}{\cos^2\theta}$. [1]

(ii) Solve the differential equation and calculate the value of x when $\theta = \frac{1}{3}\pi$, giving your answer correct to 3 significant figures. [7]

52. [9709/s18/31/q6]

In a certain chemical reaction the amount, x grams, of a substance is decreasing. The differential equation relating x and t, the time in seconds since the reaction started, is

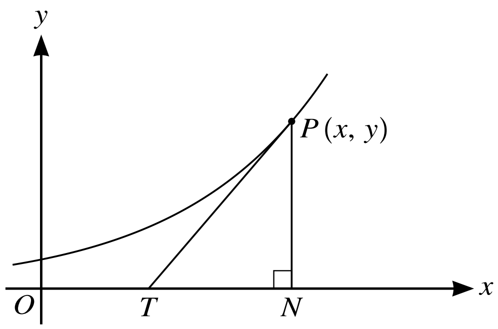
$$\frac{dx}{dt} = -kx\sqrt{t},$$

where k is a positive constant. It is given that $x = 100$ at the start of the reaction.

(i) Solve the differential equation, obtaining a relation between x, t and k. [5]

(ii) Given that $t = 25$ when $x = 80$, find the value of t when $x = 40$. [3]

53. [9709/s18/32/q3]



In the diagram, the tangent to a curve at the point P with coordinates (x, y) meets the x-axis at T. The point N is the foot of the perpendicular from P to the x-axis. The curve is such that, for all values of x, the gradient of the curve is positive and $TN = 2$.

(i) Show that the differential equation satisfied by x and y is $\frac{dy}{dx} = \frac{1}{2}y$. [1]

The point with coordinates (4, 3) lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

54. [9709/s18/33/q6]

(i) Express $\dfrac{1}{4-y^2}$ in partial fractions. [2]

(ii) The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = 4 - y^2,$$

and $y = 1$ when $x = 1$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

55. [9709/w18/31/q5]

The coordinates (x, y) of a general point on a curve satisfy the differential equation

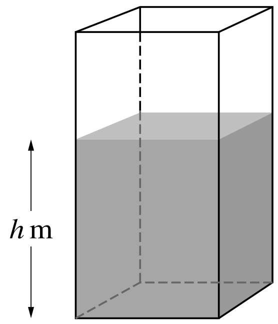
$$x\frac{dy}{dx} = (2 - x^2)y.$$

The curve passes through the point $(1, 1)$. Find the equation of the curve, obtaining an expression for y in terms of x. [7]

56. [9709/w18/32/q6]

A certain curve is such that its gradient at a general point with coordinates (x, y) is proportional to $\frac{y^2}{x}$. The curve passes through the points with coordinates $(1, 1)$ and $(\mathrm{e}, 2)$. By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [8]

57. [9709/m17/32/q7]



A water tank has vertical sides and a horizontal rectangular base, as shown in the diagram. The area of the base is $2\,\text{m}^2$. At time $t = 0$ the tank is empty and water begins to flow into it at a rate of $1\,\text{m}^3$ per hour. At the same time water begins to flow out from the base at a rate of $0.2\sqrt{h}\,\text{m}^3$ per hour, where h m is the depth of water in the tank at time t hours.

(i) Form a differential equation satisfied by h and t, and show that the time T hours taken for the depth of water to reach 4 m is given by

$$T = \int_0^4 \frac{10}{5 - \sqrt{h}}\,\mathrm{d}h. \qquad [3]$$

(ii) Using the substitution $u = 5 - \sqrt{h}$, find the value of T. [6]

58. [9709/s17/31/q9]

(i) Express $\dfrac{1}{x(2x+3)}$ in partial fractions. [2]

(ii) The variables x and y satisfy the differential equation

$$x(2x+3)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$. Solve the differential equation and calculate the value of y when $x = 9$, giving your answer correct to 3 significant figures. [7]

59. [9709/s17/32/q5]

In a certain chemical process a substance A reacts with and reduces a substance B. The masses of A and B at time t after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.2xy$ and $x = \frac{10}{(1+t)^2}$. At the beginning of the process $y = 100$.

(i) Form a differential equation in y and t, and solve this differential equation. [6]

(ii) Find the exact value approached by the mass of B as t becomes large. State what happens to the mass of A as t becomes large. [2]

60. [9709/s17/33/q8]

In a certain chemical reaction, a compound A is formed from a compound B. The masses of A and B at time t after the start of the reaction are x and y respectively and the sum of the masses is equal to 50 throughout the reaction. At any time the rate of increase of the mass of A is proportional to the mass of B at that time.

(i) Explain why $\frac{dx}{dt} = k(50 - x)$, where k is a constant. [1]

It is given that $x = 0$ when $t = 0$, and $x = 25$ when $t = 10$.

(ii) Solve the differential equation in part **(i)** and express x in terms of t. [8]

61. [9709/w17/31/q6]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = 4\cos^2 y \tan x,$$

for $0 \leqslant x < \frac{1}{2}\pi$, and $x = 0$ when $y = \frac{1}{4}\pi$. Solve this differential equation and find the value of x when $y = \frac{1}{3}\pi$. [8]

62. [9709/w17/32/q5]

The variables x and y satisfy the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = y(x+2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x. [7]

63. [9709/m16/32/q7]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = xe^{x+y},$$

and it is given that $y = 0$ when $x = 0$.

(i) Solve the differential equation and obtain an expression for y in terms of x. [7]

(ii) Explain briefly why x can only take values less than 1. [1]

64. [9709/s16/31/q4]

The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = y(1 - 2x^2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. [6]

65. [9709/s16/32/q6]

The variables x and θ satisfy the differential equation

$$(3 + \cos 2\theta)\frac{\mathrm{d}x}{\mathrm{d}\theta} = x \sin 2\theta,$$

and it is given that $x = 3$ when $\theta = \frac{1}{4}\pi$.

(i) Solve the differential equation and obtain an expression for x in terms of θ. [7]

(ii) State the least value taken by x. [1]

66. [9709/s16/33/q5]

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{-2y} \tan^2 x,$$

for $0 \leqslant x < \frac{1}{2}\pi$, and it is given that $y = 0$ when $x = 0$. Solve the differential equation and calculate the value of y when $x = \frac{1}{4}\pi$. [8]

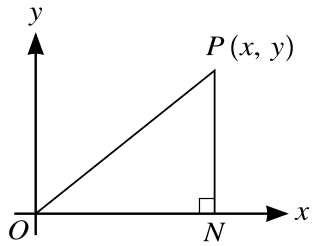
67. [9709/w16/31/q10]

A large field of area 4 km^2 is becoming infected with a soil disease. At time t years the area infected is x km^2 and the rate of growth of the infected area is given by the differential equation $\frac{dx}{dt} = kx(4 - x)$, where k is a positive constant. It is given that when $t = 0$, $x = 0.4$ and that when $t = 2$, $x = 2$.

(i) Solve the differential equation and show that $k = \frac{1}{4}\ln 3$. [9]

(ii) Find the value of t when 90% of the area of the field is infected. [2]

68. [9709/w16/33/q5]



The diagram shows a variable point P with coordinates (x, y) and the point N which is the foot of the perpendicular from P to the x-axis. P moves on a curve such that, for all $x \geqslant 0$, the gradient of the curve is equal in value to the area of the triangle OPN, where O is the origin.

(i) State a differential equation satisfied by x and y. [1]

The point with coordinates $(0, 2)$ lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

(iii) Sketch the curve. [1]

69. [9709/s15/31/q7]

Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = 4x(3y^2 + 10y + 3),$$

obtaining an expression for y in terms of x. [9]

70. [9709/s15/32/q9]

The number of organisms in a population at time t is denoted by x. Treating x as a continuous variable, the differential equation satisfied by x and t is

$$\frac{dx}{dt} = \frac{xe^{-t}}{k + e^{-t}},$$

where k is a positive constant.

(i) Given that $x = 10$ when $t = 0$, solve the differential equation, obtaining a relation between x, k and t. [6]

(ii) Given also that $x = 20$ when $t = 1$, show that $k = 1 - \frac{2}{e}$. [2]

(iii) Show that the number of organisms never reaches 48, however large t becomes. [2]

71. [9709/s15/33/q7]

The number of micro-organisms in a population at time t is denoted by M. At any time the variation in M is assumed to satisfy the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = k(\sqrt{M})\cos(0.02t),$$

where k is a constant and M is taken to be a continuous variable. It is given that when $t = 0$, $M = 100$.

(i) Solve the differential equation, obtaining a relation between M, k and t. [5]

(ii) Given also that $M = 196$ when $t = 50$, find the value of k. [2]

(iii) Obtain an expression for M in terms of t and find the least possible number of micro-organisms. [2]

72. [9709/w15/31/q8]

The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = (x+2)\sin^2 2\theta,$$

and it is given that $x = 0$ when $\theta = 0$. Solve the differential equation and calculate the value of x when $\theta = \frac{1}{4}\pi$, giving your answer correct to 3 significant figures. [9]

73. [9709/w15/33/q10]

Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time t years is denoted by N, where N is treated as a continuous variable.

(i) It is given that the rate of increase of N with respect to t is proportional to $(N - 150)$. Write down a differential equation relating N, t and a constant of proportionality. [1]

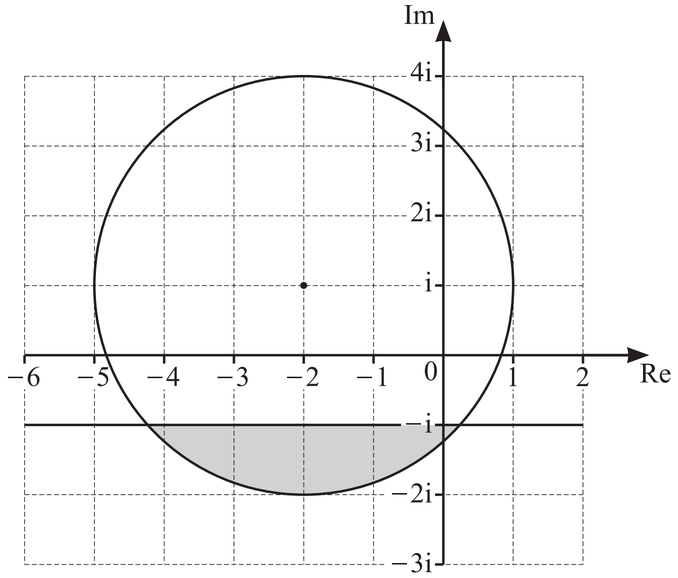
(ii) Initially, when $t = 0$, the number of plants was 650. It was noted that, at a time when there were 900 plants, the number of plants was increasing at a rate of 60 per year. Express N in terms of t. [7]

(iii) The naturalists had a target of increasing the number of plants from 650 to 2000 within 15 years. Will this target be met? [2]

Chapter 11

Complex numbers

1. [9709/m25/32/q3]



The shaded region on the Argand diagram shows points representing complex numbers z defined by two inequalities. The shaded region is bounded by a circle and a line parallel to the real axis. The boundaries of the region are included in the shaded region.

(a) Find two inequalities in terms of z that define the shaded region. [3]

....

....

....

....

....

(b) Find the greatest value of $|z|$ for points in this region. [3]

2. [9709/m25/32/q5]

The square roots of $-4+6\sqrt{5}\,\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-4+6\sqrt{5}\,\mathrm{i}$ in exact Cartesian form. [5]

3. [9709/s25/31/q3]

Find the complex numbers z for which $\frac{z+5\mathrm{i}}{z-5}$ is real and $|z| = \sqrt{17}$. Give your answers in the form $z = x + \mathrm{i}y$, where x and y are real. [6]

4. [9709/s25/31/q6]

It is given that $z_1 = 3e^{\frac{1}{4}\pi i}$, $z_2 = \frac{3}{2}e^{\frac{1}{6}\pi i}$ and $\omega = 2e^{\frac{1}{2}\pi i}$.

(a) State the values of ωz_1 and ωz_2. Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

(b) On a sketch of an Argand diagram with origin O, show the points A, B, C and D representing the complex numbers z_1, z_2, ωz_1 and ωz_2 respectively. [2]

(c) State the geometric effects of multiplying z_1 and z_2 by ω. [2]

5. [9709/s25/32/q3]

On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z-3\mathrm{i}| \leqslant 2$ and $\frac{1}{4}\pi \leqslant \arg(z-1-2\mathrm{i}) \leqslant \frac{3}{4}\pi$. [5]

6. [9709/s25/32/q5]

The square roots of $-1-4\sqrt{5}\,\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-1-4\sqrt{5}\,\mathrm{i}$ in exact Cartesian form. [5]

7. [9709/s25/33/q4]

(a) It is given that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

Show that $(z_1 z_2)^* = z_1^* z_2^*$. [3]

....

(b) $z = 3e^{\frac{1}{4}\pi i}$ is a root of the equation $z^2 + bz + c = 0$, where b and c are real.

State the other root and hence find the values of b and c. [3]

8. [9709/s25/33/q6]

Find the complex numbers z for which $\dfrac{z+4}{z+4\mathrm{i}}$ is real and $|z| = \sqrt{10}$. Give your answers in the form $z = x + \mathrm{i}y$, where x and y are real. [6]

9. [9709/s25/35/q3]

The complex numbers s and t are given by

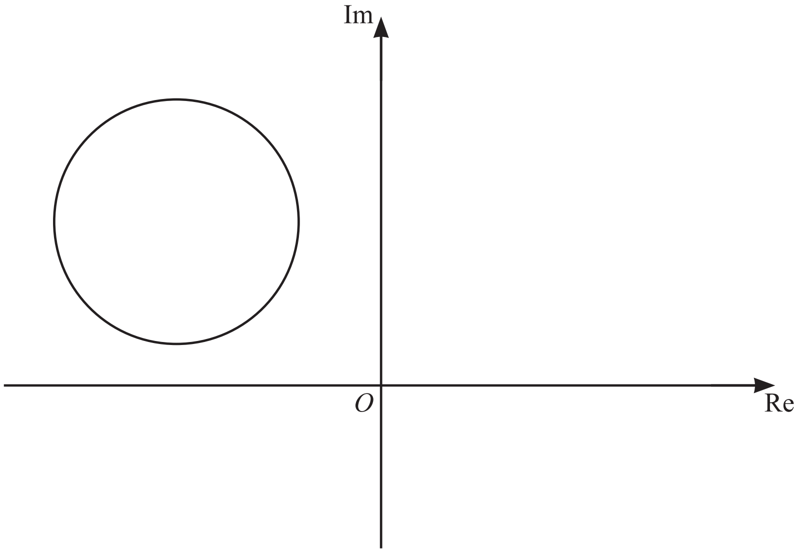
$$s = 5(\cos 0.25 + i\sin 0.25) \qquad \text{and} \qquad t = 6e^{3i}.$$

(a) Express $\frac{s}{t}$ in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$ and $r > 0$. [2]

(b) In an Argand diagram with origin O, the points A and B represent the complex numbers s and $\frac{s}{t}$ respectively.

By considering the line segments OA and OB, or otherwise, state the two geometric effects of dividing a complex number by $6e^{3i}$. [2]

10. [9709/s25/35/q5]

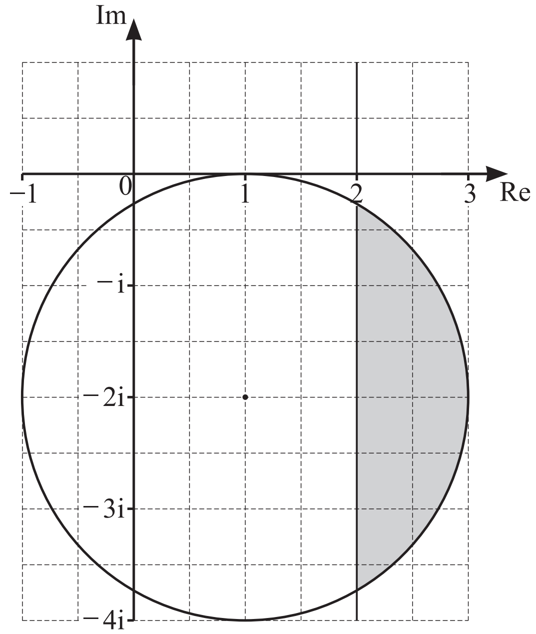


The diagram shows the locus of points representing the complex numbers, z, satisfying $|z+5-4i|=3$.

(a) For the points on this locus, determine the maximum and minimum possible values of $|z|$. [3]

(b) For the points on this locus, determine the minimum possible value of $\arg z$. [3]

11. [9709/w25/31/q5]



The shaded region on the Argand diagram shows points representing complex numbers z defined by two inequalities. The shaded region is bounded by a circle and a line parallel to the imaginary axis. The boundaries of the region are included in the shaded region.

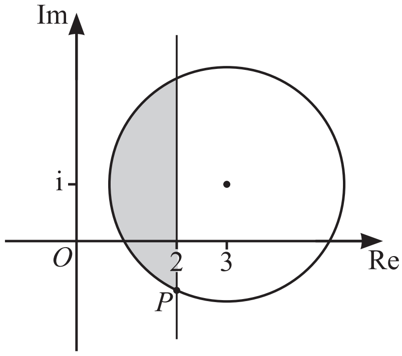
(a) Find two inequalities in terms of z that define the shaded region. [3]

(b) Calculate the least value of $\arg z$ for points in this region. [3]

12. [9709/w25/31/q6]

Solve the quadratic equation $(2+\mathrm{i})w^2+4w+2-\mathrm{i}=0$. Give your answers in the form $x+\mathrm{i}y$, where x and y are real. [5]

13. [9709/w25/32/q3]



The shaded region in the Argand diagram, bounded by a line and a circle, represents the complex numbers z satisfying

$$\operatorname{Re} z \leqslant 2 \text{ and } |z-(3+\mathrm{i})| \leqslant 2.$$

The point P shown on the diagram is one of the points of intersection of the line and the circle.

(a) Find the complex number represented by the point P. Give your answer in the form $x+\mathrm{i}y$, where x and y are real and exact. [2]

(b) Find the greatest value of $\arg z$ for points in the shaded region. [3]

14. [9709/w25/33/q4]

On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z+2\mathrm{i}| \leqslant 3$ and $|z+2\mathrm{i}| \leqslant |z-2+4\mathrm{i}|$. [5]

15. [9709/w25/33/q7]

Solve the equation $\frac{5z}{2-\mathrm{i}} - zz^* + 20 + 8\mathrm{i} = 0$. Give your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

16. [9709/w25/35/q2]

On a sketch of an Argand diagram, shade the region which represents complex numbers z satisfying both the inequalities $|z-1-3\mathrm{i}| \leqslant 2$ and $|z| \geqslant 4$. [4]

17. [9709/w25/35/q5]

Find the complex numbers, z, which satisfy the equation

$$zz^* + 5iz + 2 - 10i = 0.$$

Give your answers in the form $x + iy$, where x and y are real. [5]

18. [9709/m24/32/q3]

It is given that $z = -\sqrt{3} + \mathrm{i}$.

(a) Express z^2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

(b) The complex number ω is such that $z^2\omega$ is real and $\left|\frac{z^2}{\omega}\right| = 12$.

Find the two possible values of ω, giving your answers in the form $R\mathrm{e}^{\mathrm{i}\alpha}$, where $R > 0$ and $-\pi < \alpha \leqslant \pi$. [3]

19. [9709/m24/32/q5]

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-4-2\mathrm{i}| \leqslant 3$ and $|z| \geqslant |10-z|$. [4]

(b) Find the greatest value of $\arg z$ for points in this region. [2]

20. [9709/s24/31/q4]

The complex number u is given by $u = -1 - \mathrm{i}\sqrt{3}$.

(a) Express u in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [2]

The complex number v is given by $v = 5\left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)$.

(b) Express the complex number $\frac{v}{u}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

21. [9709/s24/31/q7]

(a) On a single Argand diagram sketch the loci given by the equations $|z-3+2i|=2$ and $|w-3+2i|=|w+3-4i|$ where z and w are complex numbers. [4]

(b) Hence find the least value of $|z-w|$ for points on these loci. Give your answer in an exact form. [2]

22. [9709/s24/32/q9]

The complex numbers z and ω are defined by $z = 1 - \mathrm{i}$ and $\omega = -3 + 3\sqrt{3}\mathrm{i}$.

(a) Express $z\omega$ in the form $a + b\mathrm{i}$, where a and b are real and in exact surd form. [1]

(b) Express z and ω in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ in each case. [4]

(c) On an Argand diagram, the points representing ω and $z\omega$ are A and B respectively.

Prove that OAB is an isosceles right-angled triangle, where O is the origin. [2]

(d) Using your answers to part **(b)**, prove that $\tan\frac{5}{12}\pi = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. [3]

23. [9709/s24/33/q3]

The square roots of $24-7\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $24-7\mathrm{i}$ in exact Cartesian form. [5]

24. [9709/s24/33/q6]

(a) On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z-4-3\mathrm{i}| \leqslant 2$ and $\arg(z-2-\mathrm{i}) \geqslant \frac{1}{3}\pi$. [5]

(b) Calculate the greatest value of $\arg z$ for points in this region. [2]

25. [9709/w24/31/q8]

(a) Given that $z = 1 + y\mathrm{i}$ and that y is a real number, express $\frac{1}{z}$ in the form $a + b\mathrm{i}$, where a and b are functions of y. [2]

(b) Show that $\left(a - \frac{1}{2}\right)^2 + b^2 = \frac{1}{4}$, where a and b are the functions of y found in part **(a)**. [3]

(c) On a single Argand diagram, sketch the loci given by the equations $\mathrm{Re}(z) = 1$ and $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, where z is a complex number. [3]

(d) The complex number z is such that $\mathrm{Re}(z) = 1$. Use your answer to part **(b)** to give a geometrical description of the locus of $\frac{1}{z}$. [1]

26. [9709/w24/32/q3]

The square roots of $6-8\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $6-8\mathrm{i}$ in exact Cartesian form. [5]

27. [9709/w24/32/q5]

(a) The complex number u is given by

$$u = \frac{\left(\cos\frac{1}{7}\pi + \mathrm{i}\sin\frac{1}{7}\pi\right)^4}{\cos\frac{1}{7}\pi - \mathrm{i}\sin\frac{1}{7}\pi}.$$

Find the exact value of arg u. [2]

(b) The complex numbers u and u^* are plotted on an Argand diagram.

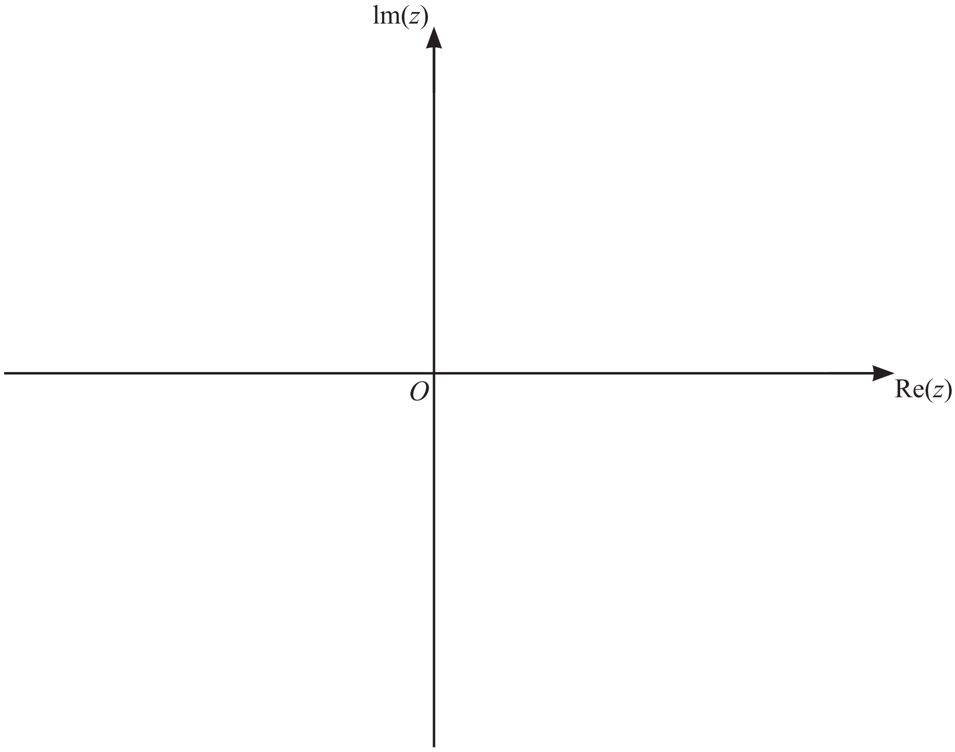
Describe the single geometrical transformation that maps u onto u^* and state the exact value of arg u^*. [2]

28. [9709/w24/33/q1]

The complex number z satisfies $|z| = 2$ and $0 \leqslant \arg z \leqslant \frac{1}{4}\pi$.

(a) On the Argand diagram below, sketch the locus of the points representing z. [2]

(b) On the **same diagram**, sketch the locus of the points representing z^2. [2]



29. [9709/w24/33/q4]

Find the complex number z satisfying the equation

$$\frac{z-3\mathrm{i}}{z+3\mathrm{i}} = \frac{2-9\mathrm{i}}{5}.$$

Give your answer in the form $x+\mathrm{i}y$, where x and y are real. [5]

30. [9709/m23/32/q2]

(a) On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leqslant \arg(z - 1 - 2\mathrm{i}) \leqslant \frac{1}{3}\pi$ and $\operatorname{Re} z \leqslant 3$. [3]

(b) Calculate the least value of $\arg z$ for points in the region from **(a)**. Give your answer in radians correct to 3 decimal places. [2]

31. [9709/m23/32/q4]

Solve the equation

$$\frac{5z}{1+2\mathrm{i}} - zz^* + 30 + 10\mathrm{i} = 0,$$

giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

32. [9709/s23/31/q10]

The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $p(x)$.

(a) Show that $(x + 3)$ is a factor of $p(x)$. [2]

(b) Show that $z = -1 + 2\sqrt{6}i$ is a root of $p(z) = 0$. [3]

(c) Hence find the complex numbers z which are roots of $p(z^2) = 0$. [7]

33. [9709/s23/32/q3]

(a) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z + 3 - 2i| = 2$. [2]

(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [2]

34. [9709/s23/32/q5]

The complex number $2 + y\mathrm{i}$ is denoted by a, where y is a real number and $y < 0$. It is given that $\mathrm{f}(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $\mathrm{f}(a)$ in terms of y. [3]

(b) Given that $\mathrm{Re}\big(\mathrm{f}(a)\big) = -20$, find $\arg a$. [3]

35. [9709/s23/33/q3]

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$ and $|z| \geqslant |z - 4\mathrm{i}|$. [4]

36. [9709/s23/33/q11]

The complex number z is defined by $z = \dfrac{5a - 2\text{i}}{3 + a\text{i}}$, where a is an integer. It is given that $\arg z = -\frac{1}{4}\pi$.

(a) Find the value of a and hence express z in the form $x + \text{i}y$, where x and y are real. [6]

(b) Express z^3 in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the simplified exact values of r and θ. [3]

37. [9709/w23/31/q2]

On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2\text{i}| \leqslant |z + 2 - \text{i}|$ and $0 \leqslant \arg(z + 1) \leqslant \frac{1}{4}\pi$. [4]

38. [9709/w23/31/q4]

The complex number u is defined by $u = \dfrac{3 + 2\mathrm{i}}{a - 5\mathrm{i}}$, where a is real.

(a) Express u in the Cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [3]

(b) Given that $\arg u = \frac{1}{4}\pi$, find the value of a. [2]

39. [9709/w23/32/q4]

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 4 - 3i| \leqslant 2$ and $\operatorname{Re} z \leqslant 3$. [4]

(b) Find the greatest value of $\arg z$ for points in this region. [2]

40. [9709/w23/32/q8]

It is given that $\frac{2 + 3a\mathrm{i}}{a + 2\mathrm{i}} = \lambda(2 - \mathrm{i})$, where a and λ are real constants.

(a) Show that $3a^2 + 4a - 4 = 0$. [4]

(b) Hence find the possible values of a and the corresponding values of λ. [3]

41. [9709/w23/33/q2]

On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 1 + 2\mathrm{i}| \leqslant |z|$ and $|z - 2| \leqslant 1$. [5]

42. [9709/w23/33/q4]

Solve the quadratic equation $(3 + \mathrm{i})w^2 - 2w + 3 - \mathrm{i} = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

43. [9709/m22/32/q2]

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2 - 3\mathrm{i}| \leqslant 2$ and $\arg z \leqslant \frac{3}{4}\pi$. [4]

44. [9709/m22/32/q6]

Find the complex numbers w which satisfy the equation $w^2 + 2\mathrm{i}w^* = 1$ and are such that $\mathrm{Re}\, w \leqslant 0$. Give your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

45. [9709/s22/31/q7]

The complex number u is defined by $u = \dfrac{\sqrt{2} - a\sqrt{2}\mathrm{i}}{1 + 2\mathrm{i}}$, where a is a positive integer.

(a) Express u in terms of a, in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

It is now given that $a = 3$.

(b) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

(c) Using your answer to part **(b)**, find the two square roots of u. Give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [3]

46. [9709/s22/32/q10]

The complex number $-1 + \sqrt{7}\mathrm{i}$ is denoted by u. It is given that u is a root of the equation

$$2x^3 + 3x^2 + 14x + k = 0,$$

where k is a real constant.

(a) Find the value of k. [3]

(b) Find the other two roots of the equation. [4]

(c) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 2$. [2]

(d) Determine the greatest value of $\arg z$ for points on this locus, giving your answer in radians. [2]

47. [9709/s22/33/q5]

The complex number $3 - \mathrm{i}$ is denoted by u.

(a) Show, on an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively.

State the type of quadrilateral formed by the points O, A, B and C. [3]

(b) Express $\dfrac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(c) By considering the argument of $\dfrac{u^*}{u}$, or otherwise, prove that $\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$. [2]

48. [9709/w22/31/q2]

On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z| \leqslant 3$, $\operatorname{Re} z \geqslant -2$ and $\frac{1}{4}\pi \leqslant \arg z \leqslant \pi$. [4]

49. [9709/w22/31/q5]

The complex numbers u and w are defined by $u = 2e^{\frac{1}{4}\pi i}$ and $w = 3e^{\frac{1}{3}\pi i}$.

(a) Find $\frac{u^2}{w}$, giving your answer in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [3]

(b) State the least positive integer n such that both $\operatorname{Im} w^n = 0$ and $\operatorname{Re} w^n > 0$. [1]

50. [9709/w22/32/q5]

(a) Solve the equation $z^2 - 6\mathrm{i}z - 12 = 0$, giving the answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(b) On a sketch of an Argand diagram with origin O, show points A and B representing the roots of the equation in part **(a)**. [1]

(c) Find the exact modulus and argument of each root. [3]

(d) Hence show that the triangle OAB is equilateral. [1]

51. [9709/w22/33/q5]

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2| \leqslant 2$ and $\text{Im}\, z \geqslant 1$. [4]

(b) Find the greatest value of $\arg z$ for points in the shaded region. [2]

52. [9709/w22/33/q6]

Solve the quadratic equation $(1 - 3\mathrm{i})z^2 - (2 + \mathrm{i})z + \mathrm{i} = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

53. [9709/m21/32/q8]

The complex numbers u and v are defined by $u = -4 + 2\mathrm{i}$ and $v = 3 + \mathrm{i}$.

(a) Find $\dfrac{u}{v}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) Hence express $\dfrac{u}{v}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ are exact. [2]

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC. [2]

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

54. [9709/s21/31/q5]

(a) Solve the equation $z^2 - 2piz - q = 0$, where p and q are real constants. [2]

In an Argand diagram with origin O, the roots of this equation are represented by the distinct points A and B.

(b) Given that A and B lie on the imaginary axis, find a relation between p and q. [2]

(c) Given instead that triangle OAB is equilateral, express q in terms of p. [3]

55. [9709/s21/32/q2]

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z+1-\mathrm{i}| \leqslant 1$ and $\arg(z-1) \leqslant \frac{3}{4}\pi$. [4]

56. [9709/s21/32/q5]

The complex number u is given by $u = 10 - 4\sqrt{6}\mathrm{i}$.

Find the two square roots of u, giving your answers in the form $a + \mathrm{i}b$, where a and b are real and exact. [5]

57. [9709/s21/33/q10]

(a) Verify that $-1 + \sqrt{2}\mathrm{i}$ is a root of the equation $z^4 + 3z^2 + 2z + 12 = 0$. [3]

(b) Find the other roots of this equation. [7]

58. [9709/w21/31/q10]

The complex number $1 + 2\mathrm{i}$ is denoted by u. The polynomial $2x^3 + ax^2 + 4x + b$, where a and b are real constants, is denoted by $\mathrm{p}(x)$. It is given that u is a root of the equation $\mathrm{p}(x) = 0$.

(a) Find the values of a and b. [4]

(b) State a second complex root of this equation. [1]

(c) Find the real factors of $\mathrm{p}(x)$. [2]

(d) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - u| \leqslant \sqrt{5}$ and $\arg z \leqslant \frac{1}{4}\pi$. [4]

(ii) Find the least value of $\operatorname{Im} z$ for points in the shaded region. Give your answer in an exact form. [1]

59. [9709/w21/32/q3]

(a) Given the complex numbers $u = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$, where a, b, c and d are real, prove that $(u + w)^* = u^* + w^*$. [2]

(b) Solve the equation $(z + 2 + \mathrm{i})^* + (2 + \mathrm{i})z = 0$, giving your answer in the form $x + \mathrm{i}y$ where x and y are real. [4]

60. [9709/w21/32/q5]

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - 2i| \leqslant 1$ and $\text{Im}\, z \geqslant 2$. [4]

(b) Find the greatest value of $\arg z$ for points in the shaded region, giving your answer in degrees. [3]

61. [9709/w21/33/q11]

The complex number $-\sqrt{3} + \mathrm{i}$ is denoted by u.

(a) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

(b) Hence show that u^6 is real and state its value. [2]

(c) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $0 \leqslant \arg(z - u) \leqslant \frac{1}{4}\pi$ and $\operatorname{Re} z \leqslant 2$. [4]

(ii) Find the greatest value of $|z|$ for points in the shaded region. Give your answer correct to 3 significant figures. [2]

62. [9709/m20/32/q10]

(a) The complex numbers v and w satisfy the equations

$$v + \mathrm{i}w = 5 \qquad \text{and} \qquad (1 + 2\mathrm{i})v - w = 3\mathrm{i}.$$

Solve the equations for v and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) **(i)** On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z - 2 - 3\mathrm{i}| = 1$. [2]

(ii) Calculate the least value of $\arg z$ for points on this locus. [2]

63. [9709/s20/31/q10]

(a) The complex number u is defined by $u = \dfrac{3\mathrm{i}}{a + 2\mathrm{i}}$, where a is real.

(i) Express u in the Cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [3]

(ii) Find the exact value of a for which $\arg u^* = \frac{1}{3}\pi$. [3]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2\mathrm{i}| \leqslant |z - 1 - \mathrm{i}|$ and $|z - 2 - \mathrm{i}| \leqslant 2$. [4]

(ii) Calculate the least value of $\arg z$ for points in this region. [2]

64. [9709/s20/32/q10]

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. The midpoint of OA is M. The point N lying on AB, between A and B, is such that $AN = 2NB$.

(a) Find a vector equation for the line through M and N. [5]

The line through M and N intersects the line through O and B at the point P.

(b) Find the position vector of P. [3]

(c) Calculate angle OPM, giving your answer in degrees. [3]

65. [9709/s20/33/q9]

(a) The complex numbers u and w are such that

$$u - w = 2\mathrm{i} \quad \text{and} \quad uw = 6.$$

Find u and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities

$$|z - 2 - 2\mathrm{i}| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad \operatorname{Re} z \leqslant 3.$$ [5]

66. [9709/w20/31/q2]

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| \geqslant 2$ and $|z - 1 + \mathrm{i}| \leqslant 1$. [4]

67. [9709/w20/31/q7]

(a) Verify that $-1 + \sqrt{5}\mathrm{i}$ is a root of the equation $2x^3 + x^2 + 6x - 18 = 0$. [3]

(b) Find the other roots of this equation. [4]

68. [9709/w20/32/q6]

The complex number u is defined by

$$u = \frac{7 + \mathrm{i}}{1 - \mathrm{i}}.$$

(a) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + \mathrm{i}$ and $1 - \mathrm{i}$ respectively. [2]

(c) By considering the arguments of $7 + \mathrm{i}$ and $1 - \mathrm{i}$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$ [3]

69. [9709/m19/32/q7]

(a) Showing all working and without using a calculator, solve the equation

$$(1+\mathrm{i})z^2 - (4+3\mathrm{i})z + 5 + \mathrm{i} = 0.$$

Give your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) The complex number u is given by

$$u = -1 - \mathrm{i}.$$

On a sketch of an Argand diagram show the point representing u. Shade the region whose points represent complex numbers satisfying the inequalities $|z| < |z - 2\mathrm{i}|$ and $\frac{1}{4}\pi < \arg(z-u) < \frac{1}{2}\pi$. [4]

70. [9709/s19/31/q10]

The complex number $(\sqrt{3}) + \mathrm{i}$ is denoted by u.

(i) Express u in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. Hence or otherwise state the exact values of the modulus and argument of u^4. [5]

(ii) Verify that u is a root of the equation $z^3 - 8z + 8\sqrt{3} = 0$ and state the other complex root of this equation. [3]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - u| \leqslant 2$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of z. [5]

71. [9709/s19/32/q5]

It is given that the complex number $-1 + (\sqrt{3})\mathrm{i}$ is a root of the equation

$$kx^3 + 5x^2 + 10x + 4 = 0,$$

where k is a real constant.

(i) Write down another root of the equation. [1]

(ii) Find the value of k and the third root of the equation. [6]

72. [9709/s19/33/q8]

The complex number u is defined by

$$u = \frac{4\mathrm{i}}{1 - (\sqrt{3})\mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(ii) Find the exact modulus and argument of u. [2]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| < 2$ and $|z - u| < |z|$. [4]

73. [9709/w19/31/q10]

(a) The complex number u is given by $u = -3 - (2\sqrt{10})\mathrm{i}$. Showing all necessary working and without using a calculator, find the square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5]

(b) On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$, $\arg z \geqslant \frac{1}{4}\pi$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of the complex number z. [5]

74. [9709/w19/32/q7]

(a) Find the complex number z satisfying the equation

$$z + \frac{iz}{z^*} - 2 = 0,$$

where z^* denotes the complex conjugate of z. Give your answer in the form $x + iy$, where x and y are real. [5]

(b) **(i)** On a single Argand diagram sketch the loci given by the equations $|z - 2i| = 2$ and $\operatorname{Im} z = 3$, where $\operatorname{Im} z$ denotes the imaginary part of z. [2]

(ii) In the first quadrant the two loci intersect at the point P. Find the exact argument of the complex number represented by P. [2]

75. [9709/w19/33/q6]

The complex number with modulus 1 and argument $\frac{1}{3}\pi$ is denoted by w.

(i) Express w in the form $x + \mathrm{i}y$, where x and y are real and exact. [1]

The complex number $1 + 2\mathrm{i}$ is denoted by u. The complex number v is such that $|v| = 2|u|$ and $\arg v = \arg u + \frac{1}{3}\pi$.

(ii) Sketch an Argand diagram showing the points representing u and v. [2]

(iii) Explain why v can be expressed as $2uw$. Hence find v, giving your answer in the form $a + \mathrm{i}b$, where a and b are real and exact. [4]

76. [9709/m18/32/q9]

The complex number $1 + 2\mathrm{i}$ is denoted by u.

(i) It is given that u is a root of the equation $2x^3 - x^2 + 4x + k = 0$, where k is a constant.

(a) Showing all working and without using a calculator, find the value of k. [3]

(b) Showing all working and without using a calculator, find the other two roots of this equation. [4]

(ii) On an Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 1$. Determine the least value of $\arg z$ for points on this locus. Give your answer in radians correct to 2 decimal places. [4]

77. [9709/s18/31/q7]

(i) Showing all working and without using a calculator, solve the equation $z^2 + (2\sqrt{6})z + 8 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) The points representing the roots are A and B, and O is the origin. Find angle AOB. [3]

(iv) Prove that triangle AOB is equilateral. [1]

78. [9709/s18/32/q7]

The complex numbers $-3\sqrt{3} + \mathrm{i}$ and $\sqrt{3} + 2\mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, where x and y are real and exact, the complex numbers uv and $\frac{u}{v}$. [5]

(ii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers u and v respectively. Prove that angle $AOB = \frac{2}{3}\pi$. [3]

79. [9709/s18/33/q9]

(a) Find the complex number z satisfying the equation

$$3z - iz^* = 1 + 5i,$$

where z^* denotes the complex conjugate of z. [4]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| \leqslant 3$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of z. Calculate the greatest value of $\arg z$ for points in this region. Give your answer in radians correct to 2 decimal places. [5]

80. [9709/w18/31/q8]

(a) Showing all necessary working, express the complex number $\frac{2+3\mathrm{i}}{1-2\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the values of r and θ correct to 3 significant figures. [5]

(b) On an Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z - 3 + 2\mathrm{i}| = 1$. Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [4]

81. [9709/w18/32/q9]

(a) **(i)** Without using a calculator, express the complex number $\frac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in the form $x+\mathrm{i}y$, where x and y are real. [2]

(ii) Hence, without using a calculator, express $\frac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in the form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and $-\pi<\theta\leqslant\pi$, giving the exact values of r and θ. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z-3\mathrm{i}|\leqslant 1$ and $\operatorname{Re} z\leqslant 0$, where $\operatorname{Re} z$ denotes the real part of z. Find the greatest value of $\arg z$ for points in this region, giving your answer in radians correct to 2 decimal places. [5]

82. [9709/m17/32/q8]

The polynomial $z^4 + 3z^2 + 6z + 10$ is denoted by $\mathrm{p}(z)$. The complex number $-1 + \mathrm{i}$ is denoted by u.

(i) Showing all your working, verify that u is a root of the equation $\mathrm{p}(z) = 0$. [3]

(ii) Find the other three roots of the equation $\mathrm{p}(z) = 0$. [7]

83. [9709/s17/31/q7]

The complex numbers u and w are defined by $u = -1 + 7\mathrm{i}$ and $w = 3 + 4\mathrm{i}$.

(i) Showing all your working, find in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers $u - 2w$ and $\frac{u}{w}$. [4]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, w and $u - 2w$ respectively.

(ii) Prove that angle $AOB = \frac{1}{4}\pi$. [2]

(iii) State fully the geometrical relation between the line segments OB and CA. [2]

84. [9709/s17/32/q6]

The complex number $2 - \mathrm{i}$ is denoted by u.

(i) It is given that u is a root of the equation $x^3 + ax^2 - 3x + b = 0$, where the constants a and b are real. Find the values of a and b. [4]

(ii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| < 1$ and $|z| < |z + \mathrm{i}|$. [4]

85. [9709/s17/33/q11]

(a) The complex numbers z and w satisfy the equations

$$z + (1 + \mathrm{i})w = \mathrm{i} \qquad \text{and} \qquad (1 - \mathrm{i})z + \mathrm{i}w = 1.$$

Solve the equations for z and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) The complex numbers u and v are given by $u = 1 + (2\sqrt{3})\mathrm{i}$ and $v = 3 + 2\mathrm{i}$. In an Argand diagram, u and v are represented by the points A and B. A third point C lies in the first quadrant and is such that $BC = 2AB$ and angle $ABC = 90°$. Find the complex number z represented by C, giving your answer in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

86. [9709/w17/31/q7]

(a) The complex number u is given by $u = 8 - 15\mathrm{i}$. Showing all necessary working, find the two square roots of u. Give answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5]

(b) On an Argand diagram, shade the region whose points represent complex numbers satisfying both the inequalities $|z - 2 - \mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z - \mathrm{i}) \leqslant \frac{1}{4}\pi$. [4]

87. [9709/w17/32/q7]

The complex number $1 - (\sqrt{3})\mathrm{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Show that $u^3 + 8 = 0$. [2]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| \leqslant 2$ and $\operatorname{Re} z \geqslant 2$, where $\operatorname{Re} z$ denotes the real part of z. [4]

88. [9709/m16/32/q10]

(a) Find the complex number z satisfying the equation $z^* + 1 = 2iz$, where z^* denotes the complex conjugate of z. Give your answer in the form $x + iy$, where x and y are real. [5]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z + 1 - 3i| \leqslant 1$ and $\operatorname{Im} z \geqslant 3$, where $\operatorname{Im} z$ denotes the imaginary part of z. [4]

(ii) Determine the difference between the greatest and least values of $\arg z$ for points lying in this region. [2]

89. [9709/s16/31/q10]

(a) Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) **(i)** On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2\mathrm{i}| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

90. [9709/s16/32/q10]

(a) Showing all necessary working, solve the equation $iz^2 + 2z - 3i = 0$, giving your answers in the form $x + iy$, where x and y are real and exact. [5]

(b) **(i)** On a sketch of an Argand diagram, show the locus representing complex numbers satisfying the equation $|z| = |z - 4 - 3i|$. [2]

(ii) Find the complex number represented by the point on the locus where $|z|$ is least. Find the modulus and argument of this complex number, giving the argument correct to 2 decimal places. [3]

91. [9709/s16/33/q9]

The complex numbers $-1 + 3i$ and $2 - i$ are denoted by u and v respectively. In an Argand diagram with origin O, the points A, B and C represent the numbers u, v and $u + v$ respectively.

(i) Sketch this diagram and state fully the geometrical relationship between OB and AC. [4]

(ii) Find, in the form $x + iy$, where x and y are real, the complex number $\frac{u}{v}$. [3]

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

92. [9709/w16/31/q9]

(a) Solve the equation $(1 + 2\mathrm{i})w^2 + 4w - (1 - 2\mathrm{i}) = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 1 - \mathrm{i}| \leqslant 2$ and $-\frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{4}\pi$. [5]

93. [9709/w16/33/q7]

The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})\mathrm{i}$. The complex conjugate of z is denoted by z^*.

(i) Find the modulus and argument of z. [2]

(ii) Express each of the following in the form $x + \mathrm{i}y$, where x and y are real and exact:

(a) $z + 2z^*$;

(b) $\dfrac{z^*}{\mathrm{i}z}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and $\mathrm{i}z$ respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

94. [9709/s15/31/q8]

The complex number w is defined by $w = \dfrac{22 + 4\mathrm{i}}{(2 - \mathrm{i})^2}$.

(i) Without using a calculator, show that $w = 2 + 4\mathrm{i}$. [3]

(ii) It is given that p is a real number such that $\frac{1}{4}\pi \leqslant \arg(w + p) \leqslant \frac{3}{4}\pi$. Find the set of possible values of p. [3]

(iii) The complex conjugate of w is denoted by w^*. The complex numbers w and w^* are represented in an Argand diagram by the points S and T respectively. Find, in the form $|z - a| = k$, the equation of the circle passing through S, T and the origin. [3]

95. [9709/s15/32/q7]

The complex number u is given by $u = -1 + (4\sqrt{3})\mathrm{i}$.

(i) Without using a calculator and showing all your working, find the two square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the real numbers a and b are exact. [5]

(ii) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the relation $|z - u| = 1$. Determine the greatest value of $\arg z$ for points on this locus. [4]

96. [9709/s15/33/q8]

The complex number $1 - \mathrm{i}$ is denoted by u.

(i) Showing your working and without using a calculator, express

$$\frac{\mathrm{i}}{u}$$

in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) On an Argand diagram, sketch the loci representing complex numbers z satisfying the equations $|z - u| = |z|$ and $|z - \mathrm{i}| = 2$. [4]

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part **(ii)**. [3]

97. [9709/w15/31/q9]

The complex number $3 - \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively. What type of quadrilateral is $OABC$? [4]

(ii) Showing your working and without using a calculator, express $\dfrac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\dfrac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]

98. [9709/w15/33/q9]

(a) It is given that $(1 + 3i)w = 2 + 4i$. Showing all necessary working, prove that the exact value of $|w^2|$ is 2 and find $\arg(w^2)$ correct to 3 significant figures. [6]

(b) On a single Argand diagram sketch the loci $|z| = 5$ and $|z - 5| = |z|$. Hence determine the complex numbers represented by points common to both loci, giving each answer in the form $re^{i\theta}$. [4]

Formula Sheet MF19

Cambridge Assessment
International Education

List MF19

List of formulae and statistical tables

Cambridge International AS & A Level
Mathematics (9709) and Further Mathematics (9231)

For use from 2020 in all papers for the above syllabuses.

CST319

2508709701

Edited by Thoridal

PURE MATHEMATICS

Mensuration

Volume of sphere = $\frac{4}{3}\pi r^3$

Surface area of sphere = $4\pi r^2$

Volume of cone or pyramid = $\frac{1}{3} \times$ base area $\times$ height

Area of curved surface of cone = $\pi r \times$ slant height

Arc length of circle $= r\theta$ (θ in radians)

Area of sector of circle $= \frac{1}{2}r^2\theta$ (θ in radians)

Algebra

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d, \qquad S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}, \qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1), \qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \ldots + b^n$$, where n is a positive integer

and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$, where n is rational and $|x| < 1$

Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1} x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Differentiation

f(*x*)	**f′(*x*)**
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
uv	$v\frac{du}{dx} + u\frac{dv}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(\boldsymbol{x})$	$\int \mathbf{f}(\boldsymbol{x})\,\mathbf{d}\boldsymbol{x}$	
x^n	$\frac{x^{n+1}}{n+1}$	$(n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x \rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	$(x > a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$	$(\lvert x \rvert < a)$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\lvert f(x)\rvert$$

Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

FURTHER PURE MATHEMATICS

Algebra

Summations:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Trigonometry

If $t = \tan\tfrac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x \equiv 1, \qquad \sinh 2x \equiv 2\sinh x \cosh x, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$	
$\sec x$	$\ln\lvert \sec x + \tan x \rvert = \ln\lvert \tan(\frac{1}{2}x + \frac{1}{4}\pi)\rvert$	$\left(\lvert x \rvert < \frac{1}{2}\pi\right)$
$\operatorname{cosec} x$	$-\ln\lvert \operatorname{cosec} x + \cot x \rvert = \ln\lvert \tan(\frac{1}{2}x)\rvert$	$(0 < x < \pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\left(\lvert x \rvert < a\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	$(x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	

MECHANICS

Uniformly accelerated motion

$$v = u + at\,, \qquad s = \tfrac{1}{2}(u+v)t\,, \qquad s = ut + \tfrac{1}{2}at^2\,, \qquad v^2 = u^2 + 2as$$

FURTHER MECHANICS

Motion of a projectile

Equation of trajectory is:

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

Elastic strings and springs

$$T = \frac{\lambda x}{l}\,, \qquad E = \frac{\lambda x^2}{2l}$$

Motion in a circle

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$$\omega^2 r \qquad \text{or} \qquad \frac{v^2}{r}$$

Centres of mass of uniform bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\frac{3}{8}r$ from centre

Hemispherical shell of radius r: $\frac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\dfrac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\dfrac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\frac{3}{4}h$ from vertex

PROBABILITY & STATISTICS

Summary statistics

For ungrouped data:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

For grouped data:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

Discrete random variables

$$\mathrm{E}(X) = \Sigma xp, \qquad \mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2$$

For the binomial distribution $\mathrm{B}(n, p)$:

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \qquad \mu = np, \qquad \sigma^2 = np(1-p)$$

For the geometric distribution Geo(p):

$$p_r = p(1-p)^{r-1}, \qquad \mu = \frac{1}{p}$$

For the Poisson distribution $\mathrm{Po}(\lambda)$

$$p_r = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!}, \qquad \mu = \lambda, \qquad \sigma^2 = \lambda$$

Continuous random variables

$$\mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{Var}(X) = \int x^2\,\mathrm{f}(x)\,\mathrm{d}x - \{\mathrm{E}(X)\}^2$$

Sampling and testing

Unbiased estimators:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1} = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Central Limit Theorem:

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$\mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$$

FURTHER PROBABILITY & STATISTICS

Sampling and testing

Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \overline{x}_1)^2 + \Sigma(x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$$

Probability generating functions

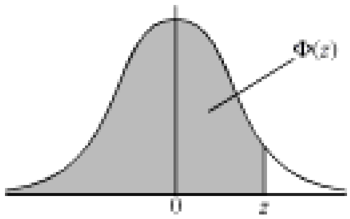
$$\mathrm{G}_X(t) = \mathrm{E}(t^X), \qquad \mathrm{E}(X) = \mathrm{G}'_X(1), \qquad \mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \{\mathrm{G}'_X(1)\}^2$$

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1, then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = \mathrm{P}(Z \leqslant z).$$

For negative values of z, use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	ADD								
											1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1, then, for each value of p, the table gives the value of z such that

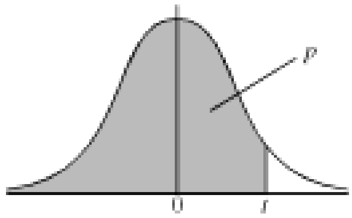
$$\mathrm{P}(Z \leqslant z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE *t*-DISTRIBUTION

If T has a t-distribution with ν degrees of freedom, then, for each pair of values of p and ν, the table gives the value of t such that:

$P(T \leqslant t) = p$.

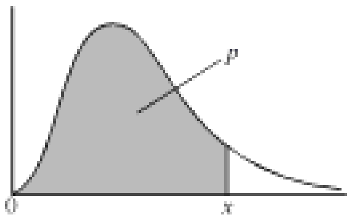


p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
$\nu = 1$	1.000	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.894	6.869
6	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.689
28	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.660
30	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE χ^2-DISTRIBUTION

If X has a χ^2-distribution with ν degrees of freedom then, for each pair of values of p and ν, the table gives the value of x such that

$$P(X \leqslant x) = p.$$



p	0.01	0.025	0.05	0.9	0.95	0.975	0.99	0.995	0.999
$\nu = 1$	$0.0^{3}1571$	$0.0^{3}9821$	$0.0^{2}3932$	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

WILCOXON SIGNED-RANK TEST

The sample has size n.

P is the sum of the ranks corresponding to the positive differences.

Q is the sum of the ranks corresponding to the negative differences.

T is the smaller of P and Q.

For each value of n the table gives the **largest** value of T which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *T*

	Level of significance			
One-tailed	0.05	0.025	0.01	0.005
Two-tailed	0.1	0.05	0.02	0.01
$n = 6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

For larger values of n, each of P and Q can be approximated by the normal distribution with mean $\frac{1}{4}n(n+1)$ and variance $\frac{1}{24}n(n+1)(2n+1)$.

WILCOXON RANK-SUM TEST

The two samples have sizes m and n, where $m \leqslant n$.

R_m is the sum of the ranks of the items in the sample of size m.

W is the smaller of R_m and $m(n + m + 1) - R_m$.

For each pair of values of m and n, the table gives the **largest** value of W which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *W*

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 3$			$m = 4$			$m = 5$			$m = 6$		
3	6	–	–									
4	6	–	–	11	10	–						
5	7	6	–	12	11	10	19	17	16			
6	8	7	–	13	12	11	20	18	17	28	26	24
7	8	7	6	14	13	11	21	20	18	29	27	25
8	9	8	6	15	14	12	23	21	19	31	29	27
9	10	8	7	16	14	13	24	22	20	33	31	28
10	10	9	7	17	15	13	26	23	21	35	32	29

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 7$			$m = 8$			$m = 9$			$m = 10$		
7	39	36	34									
8	41	38	35	51	49	45						
9	43	40	37	54	51	47	66	62	59			
10	45	42	39	56	53	49	69	65	61	82	78	74

For larger values of m and n, the normal distribution with mean $\frac{1}{2}m(m+n+1)$ and variance $\frac{1}{12}mn(m+n+1)$ should be used as an approximation to the distribution of R_m.

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Syllabus 26-27 Pure Mathematics 3

3 Pure Mathematics 3 (for Paper 3)

Knowledge of the content of Paper 1: Pure Mathematics 1 is assumed, and candidates may be required to demonstrate such knowledge in answering questions.

3.1 Algebra

Candidates should be able to:	Notes and examples
• understand the meaning of $\|x\|$, sketch the graph of $y = \|ax + b\|$ and use relations such as $\|a\| = \|b\| \Leftrightarrow a^2 = b^2$ and	Graphs of $y = \|f(x)\|$ and $y = f(\|x\|)$ for non-linear functions f are not included.
$\|x - a\| < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities	e.g. $\|3x - 2\| = \|2x + 7\|$, $2x + 5 < \|x + 1\|$.
• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)	
• use the factor theorem and the remainder theorem	e.g. to find factors and remainders, solve polynomial equations or evaluate unknown coefficients. Including factors of the form $(ax + b)$ in which the coefficient of x is not unity, and including calculation of remainders.
• recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than – $(ax + b)(cx + d)(ex + f)$ – $(ax + b)(cx + d)^2$ – $(ax + b)(cx^2 + d)$	Excluding cases where the degree of the numerator exceeds that of the denominator
• use the expansion of $(1 + x)^n$, where n is a rational number and $\|x\| < 1$.	Finding the general term in an expansion is not included. Adapting the standard series to expand e.g. $\left(2 - \frac{1}{2}x\right)^{-1}$ is included, and determining the set of values of x for which the expansion is valid in such cases is also included.

3 Pure Mathematics 3

3.2 Logarithmic and exponential functions

Candidates should be able to:	Notes and examples
• understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base)	
• understand the definition and properties of e^x and $\ln x$, including their relationship as inverse functions and their graphs	Including knowledge of the graph of $y = \mathrm{e}^{kx}$ for both positive and negative values of k.
• use logarithms to solve equations and inequalities in which the unknown appears in indices	e.g. $2^x < 5$, $3 \times 2^{3x-1} < 5$, $3^{x+1} = 4^{2x-1}$.
• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.	e.g. $y = kx^n$ gives $\ln y = \ln k + n \ln x$ which is linear in $\ln x$ and $\ln y$. $y = k(a^x)$ gives $\ln y = \ln k + x \ln a$ which is linear in x and $\ln y$.

3.3 Trigonometry

Candidates should be able to:	Notes and examples
• understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude	
• use trigonometrical identities for the simplification and exact evaluation of expressions, and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of – $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$ – the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ – the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$ – the expression of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$.	e.g. simplifying $\cos(x - 30°) - 3\sin(x - 60°)$. e.g. solving $\tan\theta + \cot\theta = 4$, $2\sec^2\theta - \tan\theta = 5$, $3\cos\theta + 2\sin\theta = 1$.

3 Pure Mathematics 3

3.4 Differentiation

Candidates should be able to:	Notes and examples
• use the derivatives of e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\tan^{-1} x$, together with constant multiples, sums, differences and composites	Derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are not required.
• differentiate products and quotients	e.g. $\frac{2x-4}{3x+2}$, $x^2 \ln x$, xe^{1-x^2}.
• find and use the first derivative of a function which is defined parametrically or implicitly.	e.g. $x = t - e^{2t}$, $y = t + e^{2t}$. e.g. $x^2 + y^2 = xy + 7$. Including use in problems involving tangents and normals.

3.5 Integration

Candidates should be able to:	Notes and examples
• extend the idea of 'reverse differentiation' to include the integration of e^{ax+b}, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$ and $\frac{1}{x^2+a^2}$	Including examples such as $\frac{1}{2+3x^2}$.
• use trigonometrical relationships in carrying out integration	e.g. use of double-angle formulae to integrate $\sin^2 x$ or $\cos^2(2x)$.
• integrate rational functions by means of decomposition into partial fractions	Restricted to types of partial fractions as specified in topic 3.1 above.
• recognise an integrand of the form $\frac{kf'(x)}{f(x)}$, and integrate such functions	e.g. integration of $\frac{x}{x^2+1}$, $\tan x$.
• recognise when an integrand can usefully be regarded as a product, and use integration by parts	e.g. integration of $x\sin 2x$, x^2e^{-x}, $\ln x$, $x\tan^{-1} x$.
• use a given substitution to simplify and evaluate either a definite or an indefinite integral.	e.g. to integrate $\sin^2 2x \cos x$ using the substitution $u = \sin x$.

3 Pure Mathematics 3

3.6 Numerical solution of equations

Candidates should be able to:	Notes and examples
• locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change	e.g. finding a pair of consecutive integers between which a root lies.
• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation	
• understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine a root to a prescribed degree of accuracy.	Knowledge of the condition for convergence is not included, but an understanding that an iteration may fail to converge is expected.

3.7 Vectors

Candidates should be able to:	Notes and examples
• use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$	
• carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms	e.g. '$OABC$ is a parallelogram' is equivalent to $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$. The general form of the ratio theorem is not included, but understanding that the midpoint of AB has position vector $\frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$ is expected.
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors	In 2 or 3 dimensions.
• understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, and find the equation of a line, given sufficient information	e.g. finding the equation of a line given the position vector of a point on the line and a direction vector, or the position vectors of two points on the line.
• determine whether two lines are parallel, intersect or are skew, and find the point of intersection of two lines when it exists	Calculation of the shortest distance between two skew lines is not required. Finding the equation of the common perpendicular to two skew lines is also not required.
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points.	e.g. finding the angle between two lines, and finding the foot of the perpendicular from a point to a line; questions may involve 3D objects such as cuboids, tetrahedra (pyramids), etc. Knowledge of the vector product is not required.

3 Pure Mathematics 3

3.8 Differential equations

Candidates should be able to:	Notes and examples
• formulate a simple statement involving a rate of change as a differential equation	The introduction and evaluation of a constant of proportionality, where necessary, is included.
• find by integration a general form of solution for a first order differential equation in which the variables are separable	Including any of the integration techniques from topic 3.5 above.
• use an initial condition to find a particular solution	
• interpret the solution of a differential equation in the context of a problem being modelled by the equation.	Where a differential equation is used to model a 'real-life' situation, no specialised knowledge of the context will be required.

3.9 Complex numbers

Candidates should be able to:	Notes and examples
• understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal	Notations $\mathrm{Re}\,z$, $\mathrm{Im}\,z$, $\|z\|$, $\arg z$, z^* should be known. The argument of a complex number will usually refer to an angle θ such that $-\pi < \theta \leqslant \pi$, but in some cases the interval $0 \leqslant \theta < 2\pi$ may be more convenient. Answers may use either interval unless the question specifies otherwise.
• carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in Cartesian form $x + \mathrm{i}y$	For calculations involving multiplication or division, full details of the working should be shown.
• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs	e.g. in solving a cubic or quartic equation where one complex root is given.
• represent complex numbers geometrically by means of an Argand diagram	
• carry out operations of multiplication and division of two complex numbers expressed in polar form $r(\cos\theta + \mathrm{i}\sin\theta) \equiv r\mathrm{e}^{\mathrm{i}\theta}$	Including the results $\|z_1 z_2\| = \|z_1\|\|z_2\|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, and corresponding results for division.
• find the two square roots of a complex number	e.g. the square roots of $5 + 12\mathrm{i}$ in exact Cartesian form. Full details of the working should be shown.
• understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers	
• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram	e.g. $\|z - a\| < k$, $\|z - a\| = \|z - b\|$, $\arg(z - a) = \alpha$.