The Question Bank of

Further Pure Mathematics 2

for CAIE 9231 paper 2.

v1.0

Edited by Thoridal

Instructions for Use

- This question bank is organized by chapter for systematic revision.
- This question bank is complied based on the 26-27 CAIE Further Pure Mathematics 2 syllabus, which is included as appendix.
- Each question includes its source for reference.
- Mark schemes are provided in the separate answer booklet.
- The formula sheet (MF19) is included as appendix.
- Use this resource for targeted practice and exam preparation.

Edited by Y.Liu

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Chapter 1

Hyberbolic functions

1. [9231/w25/24/q2]

(a) Find the exact values of x for which $\cosh 2x = 6\sinh^2 x$, giving your answers in logarithmic form. [4]

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(b) Sketch the curves $C_1 : y = \cosh 2x$ and $C_2 : y = 6\sinh^2 x$ on the same diagram. [2]

2. [9231/s23/23/q5]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\cosh^2 x = \cosh 2x + 1.$$ [3]

(b) Find the solution of the differential equation

$$\frac{dy}{dx} + 2y\tanh x = 1$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. [8]

3. **[9231/w23/21/q6]**

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\sinh 2x = 2\sinh x \cosh x.$$ [3]

(b) Using the substitution $u = \sinh x$, find $\int \sinh^2 2x \cosh x \, dx$. [4]

(c) Find the particular solution of the differential equation

$$\frac{dy}{dx} + y \tanh x = \sinh^2 2x,$$

given that $y = 4$ when $x = 0$. Give your answer in the form $y = f(x)$. [7]

4. [9231/w23/22/q7]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\sinh^2 A = \cosh 2A - 1.$$ [3]

(b) A curve has equation $y = x^2$, for $0 \leqslant x \leqslant \frac{2}{3}$. The area of the surface generated when the curve is rotated through 2π radians about the x-axis is denoted by S.

Use the substitution $x = \frac{1}{2}\sinh u$ to show that $S = \frac{1}{32}\pi\left(\frac{820}{81} - \ln 3\right)$. [9]

5. [9231/s22/21/q2]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh 2x = 2\sinh^2 x + 1.$$ [3]

(b) Find the set of values of k for which $\cosh 2x = k\sinh x$ has two distinct real roots. [5]

6. [9231/s22/23/q4]

It is given that

$$x = -t + \tan^{-1} t \qquad \text{and} \qquad y = t + \sinh^{-1} t.$$

(a) Show that $\dfrac{dy}{dx} = -\dfrac{t^2 + 1 + \sqrt{t^2 + 1}}{t^2}$. [4]

(b) Find the value of $\dfrac{d^2y}{dx^2}$ when $t = \frac{3}{4}$. [5]

7. [9231/w22/21/q4]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$ [3]

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}(\sinh x)\right) = \operatorname{sech} x$. [3]

(c) Sketch the graph of $y = \operatorname{sech} x$, stating the equation of the asymptote. [2]

(d) By considering a suitable set of n rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{n} \operatorname{sech} r < \tan^{-1}(\sinh n).$$ [3]

(e) Hence state an upper bound, in terms of π, for $\sum_{r=1}^{\infty} \operatorname{sech} r$. [1]

8. [9231/w22/22/q8]

It is given that $y = \cosh u$, where $u > 0$, and

$$\sqrt{\cosh^2 u - 1}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x}\right) + \cosh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 2\cosh u = 4\mathrm{e}^{-x}.$$

(a) Show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 4\mathrm{e}^{-x}.$$ [4]

(b) Find u in terms of x, given that, when $x = 0$, $u = \ln 3$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$. [10]

9. [9231/s21/21/q7]

(a) It is given that $y = \text{sech}^{-1}\left(x+\frac{1}{2}\right)$.

Express $\cosh y$ in terms of x and hence show that $\sinh y \frac{dy}{dx} = -\frac{1}{\left(x+\frac{1}{2}\right)^2}$. [3]

(b) Find the first three terms in the Maclaurin's series for $\text{sech}^{-1}\left(x+\frac{1}{2}\right)$ in the form

$$\ln a + bx + cx^2,$$

where a, b and c are constants to be determined. [7]

10. [9231/s21/21/q8]

The curve C has parametric equations

$$x = 2\cosh t, \quad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t, \quad \text{for } 0 \leqslant t \leqslant 1.$$

(a) Find $\dfrac{dx}{dt}$ and show that $\dfrac{dy}{dt} = 1 - \sinh^2 t$. [3]

The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) (i) Show that $A = \pi \displaystyle\int_0^1 \left(\tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt$. [4]

(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]

11. [9231/s21/23/q6]

(a) Starting from the definitions of sinh and cosh in terms of exponentials, prove that

$$2\sinh^2 x = \cosh 2x - 1.$$ [3]

(b) Find the solution to the differential equation

$$\frac{dy}{dx} + y\coth x = 4\sinh x$$

for which $y = 1$ when $x = \ln 3$. [7]

12. [9231/w21/21/q8]

(a) Starting from the definition of cosh in terms of exponentials, prove that

$$2\cosh^2 A = \cosh 2A + 1. \qquad [3]$$

The curve C has parametric equations

$$x = 2\cosh 2t + 3t, \quad y = \tfrac{3}{2}\cosh 2t - 4t, \quad \text{for } -\tfrac{1}{2} \leqslant t \leqslant \tfrac{1}{2}.$$

The area of the surface generated when C is rotated through 2π radians about the y-axis is denoted by A.

(b) **(i)** Show that $A = 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)\cosh 2t \, dt$. [4]

(ii) Hence find A in terms of π and e. [7]

13. [9231/w21/22/q1]

It is given that $y = \sinh(x^2) + \cosh(x^2)$.

(a) Use standard results from the list of formulae (MF19) to find the Maclaurin's series for y in terms of x up to and including the term in x^4. [2]

(b) Deduce the value of $\dfrac{d^4y}{dx^4}$ when $x = 0$. [1]

(c) Use your answer to part **(a)** to find an approximation to $\int_0^{\frac{1}{2}} y \, dx$, giving your answer as a rational fraction in its lowest terms. [2]

14. [9231/w21/22/q8]

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2 x = \operatorname{sech}^2 x.$$ [3]

(b) Using the substitution $u = \tanh x$, or otherwise, find $\int \operatorname{sech}^2 x \tanh^2 x \, dx$. [2]

It is given that, for $n \geqslant 0$, $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$.

(c) Show that, for $n \geqslant 2$,

$$(n+1)I_n = \left(\tfrac{4}{5}\right)^3 \left(\tfrac{3}{5}\right)^{n-2} + (n-2)I_{n-2}.$$ [5]

[You may use the result that $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.]

(d) Find the value of I_4. [3]

15. [9231/s20/21/q5]

The curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$ intersect at the point where $x = a$.

(a) Find the exact value of a, giving your answer in logarithmic form. [4]

(b) Sketch C_1 and C_2 on the same diagram. [2]

(c) Find the exact value of the length of the arc of C_1 from $x = 0$ to $x = a$. [5]

16. [9231/s20/23/q6]

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta.$$ [3]

The variables x and y are such that $\tanh y = \cos\left(x + \frac{1}{4}\pi\right)$, for $-\frac{1}{4}\pi < x < \frac{3}{4}\pi$.

(b) By differentiating the equation $\tanh y = \cos\left(x + \frac{1}{4}\pi\right)$ with respect to x, show that

$$\frac{dy}{dx} = -\operatorname{cosec}\left(x + \frac{1}{4}\pi\right).$$ [4]

(c) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\cos\left(x + \frac{1}{4}\pi\right)\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [5]

17. [9231/w20/21/q5]

It is given that

$$x = \sinh^{-1} t, \quad y = \cos^{-1} t,$$

where $-1 < t < 1$.

(a) By differentiating $\cos y$ with respect to t, show that $\frac{dy}{dt} = -\frac{1}{\sqrt{1-t^2}}$. [4]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [5]

18. [9231/w20/21/q8]

(a) Sketch the graph of $y = \coth x$ for $x > 0$ and state the equations of the asymptotes. [2]

(b) Starting from the definitions of coth and cosech in terms of exponentials, prove that

$$\coth^2 x - \operatorname{cosech}^2 x = 1.$$ [3]

The curve C has equation $y = \ln\coth\left(\frac{1}{2}x\right)$ for $x > 0$.

(c) Show that $\dfrac{dy}{dx} = -\operatorname{cosech} x$. [3]

(d) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where a is a positive constant.

Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a. [7]

19. [9231/w20/22/q2]

A curve has equation $y = \cosh x$, for $0 \leqslant x \leqslant \frac{1}{2}$.

Find, in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis. [6]

Chapter 2

Matrices

1. [9231/s25/21/q8]

(a) It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^3$ with corresponding eigenvalue λ^3. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \end{pmatrix}.$$

(b) Show that the eigenvalues of $\mathbf{A}$ are -1, 1 and 5. [2]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} - 2\mathbf{I} = \mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of $\mathbf{A}$ to show that $(\mathbf{A} - 2\mathbf{I})^3 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$ where a, b and c are constants to be determined. [3]

2. [9231/s25/23/q8]

(a) Find the values of a for which the system of equations

$$\begin{aligned} \tfrac{3}{2}x + 3y + 8z &= 1, \\ ax + 3y + 4z &= 2, \\ ay - z &= 3, \end{aligned}$$

does not have a unique solution. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 3 & 8 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) Given that $\mathbf{B} = \mathbf{A}^{-1}$, use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{B}^2 = p\mathbf{I} + q\mathbf{A}$, where p and q are constants to be determined. [4]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

3. [9231/s25/24/q1]

(a) Find the values of k for which the system of equations

$$\begin{aligned} x + 2y + 3z &= 1, \\ kx + 5y + 6z &= 2, \\ 7x + 2ky + 9z &= 3, \end{aligned}$$

does not have a unique solution. [3]

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(b) Given that $k = 1$, show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

4. [9231/s25/24/q7]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 11 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^6 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of **A** to show that

$$\mathbf{A}^6 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I},$$

where a, b and c are integers to be determined. [4]

5. [9231/w25/21/q1]

Find the values of k for which the system of equations

$$\begin{aligned} x - y + z &= 0, \\ x + ky + 3z &= 0, \\ x + 2y + kz &= 0, \end{aligned}$$

does not have a unique solution. [3]

6. [9231/w25/21/q6]

The matrix **P** has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

(a) State, in terms of a, the eigenvalues of **P**. [1]

(b) **(i)** Find $\mathbf{P}^2$ in terms of a. [1]

(ii) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$ in terms of a. [3]

The 3×3 matrix **A** has eigenvalues 1, 2, 3 with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

(c) Find **A** in terms of a. [5]

7. [9231/w25/22/q8]

(a) Find the values of k for which the system of equations

$$\begin{aligned} x - y + 2kz &= 1, \\ kx + y + 2z &= 2, \\ 2x - y + z &= 3, \end{aligned}$$

does not have a unique solution. [3]

(b) Given that $k = -\frac{1}{2}$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [3]

(c) Given instead that $k = -1$, show that the system of equations in part **(a)** is also inconsistent. Interpret this situation geometrically. [4]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}.$$

(d) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A}$ where p and q are integers to be determined. [5]

8. [9231/w25/24/q8]

(a) Find the set of values of a for which the system of equations

$$\begin{aligned} 5x + ay \quad &= 3, \\ 20x - 5y \quad &= 2, \\ 2x - 3y + z &= 1, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [3]

(b) Given that $a = -\frac{5}{4}$, show that the system of equations in part **(a)** is inconsistent and interpret this situation geometrically. [3]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{pmatrix}.$$

(c) Find a matrix **P** and a diagonal matrix **D** such that $(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{PDP}^{-1}$. [7]

9. [9231/s24/21/q8]

(a) Find the set of values of a for which the system of equations

$$\begin{aligned} 6x + ay &= 3, \\ 2x - y &= 1, \\ x + 5y + 4z &= 2 \end{aligned}$$

has a unique solution. [2]

(b) Show that the system of equations in part **(a)** is consistent for all values of a. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 6 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 5 & 4 \end{pmatrix}.$$

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(14\mathbf{A} + 24\mathbf{I})^2 = \mathbf{PDP}^{-1}$. [7]

(d) Use the characteristic equation of $\mathbf{A}$ to show that

$$(14\mathbf{A} + 24\mathbf{I})^2 = \mathbf{A}^4(\mathbf{A} + b\mathbf{I})^2,$$

where b is an integer to be determined. [4]

10. [9231/s24/23/q8]

The planes Π_1 and Π_2 do not intersect and are both perpendicular to $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. The line l intersects Π_1 at the point $(1,6,0)$ and intersects Π_2 at the point $(3,-6,0)$.

(a) Find Cartesian equations of Π_1 and Π_2. [3]

(b) Express the vector equation of l in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors to be determined, and hence show that for points on l, $\frac{1}{2}x+\frac{1}{12}y = 1$ and $z = 0$. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{12} & 0 \end{pmatrix}.$$

(c) Show that the characteristic equation of $\mathbf{A}$ is $-\lambda^3+3\lambda^2+\frac{7}{4}\lambda = 0$ and hence find the eigenvalues of $\mathbf{A}$. [3]

(d) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^n = \mathbf{PDP}^{-1}$, where n is a positive integer. [6]

11. [9231/w24/21/q1]

Find the set of values of k for which the system of equations

$$\begin{aligned} x+5y+6z &= 1, \\ kx+2y+2z &= 2, \\ -3x+4y+8z &= 3, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [4]

12. [9231/w24/21/q4]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}.$$

(a) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and state the corresponding eigenvalue. [2]

(b) Show that the characteristic equation of **A** is $\lambda^3 - 19\lambda - 30 = 0$ and hence find the other eigenvalues of **A**. [3]

(c) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. [4]

13. [9231/w24/22/q8]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 7 & 9 \\ 4 & 1 & 7 \end{pmatrix}.$$

(a) Show that the characteristic equation of **A** is $\lambda^3 - 12\lambda^2 + 12\lambda + 80 = 0$ and find the eigenvalues of **A**. [4]

(b) Use the characteristic equation of **A** to show that

$$\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I},$$

where p, q and r are integers to be determined. [4]

(c) Find a matrix **P** and a diagonal matrix **D** such that $(\mathbf{A} - 3\mathbf{I})^4 = \mathbf{PDP}^{-1}$. [6]

14. [9231/s23/21/q1]

(a) Show that the system of equations

$$\begin{aligned} x+2y+3z &= 1, \\ 4x+5y+6z &= 1, \\ 7x+8y+9z &= 1, \end{aligned}$$

does not have a unique solution. [2]

(b) Show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

15. [9231/s23/21/q5]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 18 & 5 & -11 \\ 8 & 6 & -4 \\ 32 & 10 & -20 \end{pmatrix}.$$

(a) Show that the characteristic equation of **A** is $\lambda^3 - 4\lambda^2 - 20\lambda + 48 = 0$ and hence find the eigenvalues of **A**. [4]

(b) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [6]

16. [9231/s23/23/q8]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} a & -6a & 2a+2 \\ 0 & 1-a & 0 \\ 0 & 2-a & -1 \end{pmatrix}$$

where a is a constant with $a \neq 0$ and $a \neq 1$.

(a) Show that the equation $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has a unique solution and interpret this situation geometrically. [3]

(b) Show that the eigenvalues of **A** are a, $1-a$ and -1. [2]

(c) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^4 = \mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of **A** to find $\mathbf{A}^4$ in terms of **A** and a. [3]

17. [9231/w23/21/q1]

Show that the system of equations

$$\begin{aligned} 14x - 4y + 6z &= 5, \\ x + y + kz &= 3, \\ -21x + 6y - 9z &= 14, \end{aligned}$$

where k is a constant, does not have a unique solution and interpret this situation geometrically. [4]

18. [9231/w23/21/q7]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -6 & 2 & 13 \\ 0 & -2 & 5 \\ 0 & 0 & 8 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. [7]

(b) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. [4]

19. [9231/w23/22/q6]

The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) State the eigenvalues of **P**. [1]

(b) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$. [4]

The 3×3 matrix **A** has distinct non-zero eigenvalues a, $\frac{1}{2}$, 2 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

respectively.

(c) Find $\mathbf{A}^{-1}$ in terms of a. [5]

20. [9231/s22/21/q8]

(a) Find the value of a for which the system of equations

$$\begin{aligned} 3x + ay \quad &= 0, \\ 5x - \ y \quad &= 0, \\ x + 3y + 2z &= 0, \end{aligned}$$

does not have a unique solution. [2]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 5 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

(b) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

(c) Use the characteristic equation of **A** to show that

$$(\mathbf{A} + 6\mathbf{I})^2 = \mathbf{A}^4(\mathbf{A} + b\mathbf{I})^2,$$

where b is an integer to be determined. [4]

21. [9231/s22/23/q3]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -9 & 5 \\ 5 & -8 & 5 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of **A**. [4]

(b) Use the characteristic equation of **A** to show that $\mathbf{A}^{-1} = p\mathbf{A}^2 + q\mathbf{I}$, where p and q are constants to be determined. [3]

22. [9231/w22/21/q2]

(a) Show that the system of equations

$$x - y + 2z = 4,$$
$$x - y - 3z = a,$$
$$x - y + 7z = 13,$$

where a is a constant, does not have a unique solution. [2]

(b) Given that $a = -5$, show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

(c) Given instead that $a \neq -5$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [2]

23. [9231/w22/21/q6]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -7 \\ 0 & 5 & 7 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of **A** to show that

$$\mathbf{A}^4 = a\mathbf{A}^2 + b\mathbf{I},$$

where a and b are integers to be determined. [4]

24. [9231/w22/22/q1]

(a) Find the set of values of k for which the system of equations

$$x + 2y + 3z = 1,$$
$$kx + 4y + 6z = 0,$$
$$7x + 8y + 9z = 3,$$

has a unique solution. [3]

(b) Interpret the situation geometrically in the case where the system of equations does not have a unique solution. [2]

25. [9231/w22/22/q7]

(a) It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

Show that λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ for which $\mathbf{e}$ is a corresponding eigenvector. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ 15 & -4 & 3 \\ 3 & 0 & 2 \end{pmatrix}.$$

(b) Given that -1 is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector. [2]

(c) It is also given that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$. Find the corresponding eigenvalues. [2]

(d) Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [2]

(e) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^{-1} = p\mathbf{A}^2 + q\mathbf{I}$, where p and q are rational numbers to be determined. [4]

26. [9231/s21/21/q1]

(a) Given that a is an integer, show that the system of equations

$$\begin{aligned} ax + 3y + z &= 14, \\ 2x + y + 3z &= 0, \\ -x + 2y - 5z &= 17, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [4]

(b) Find the value of a for which $x = 1,\ y = 4,\ z = -2$ is the solution to the system of equations in part **(a)**. [1]

27. [9231/s21/21/q6]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of **A** to find $\mathbf{A}^3$. [4]

28. [9231/s21/23/q8]

(a) Find the value of a for which the system of equations

$$\begin{aligned} 13x + 18y - 28z &= 0, \\ -4x - ay + 8z &= 0, \\ 2x + 6y - 5z &= 0, \end{aligned}$$

does not have a unique solution. [2]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}.$$

(b) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. [1]

(c) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [8]

(d) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$ in terms of **A**. [2]

29. [9231/w21/21/q2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Use the characteristic equation of $\mathbf{A}$ to show that

$$\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{I},$$

where p and q are integers to be determined. [6]

30. [9231/w21/22/q6]

The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 6 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$. [5]

(b) Find the matrix **A** such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$ [4]

(c) State the eigenvalues and corresponding eigenvectors of $\mathbf{A}^3$. [2]

31. [9231/s20/21/q8]

(a) Find the values of a for which the system of equations

$$\begin{aligned} 3x + y + z &= 0, \\ ax + 6y - z &= 0, \\ ay - 2z &= 0, \end{aligned}$$

does not have a unique solution. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Use the characteristic equation of $\mathbf{A}$ to find the inverse of $\mathbf{A}^2$. [4]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [7]

32. [9231/s20/23/q3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of $\mathbf{A}$. [4]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

33. [9231/w20/21/q3]

(a) Show that the system of equations

$$x-2y-4z=1,$$
$$x-2y+kz=1,$$
$$-x+2y+2z=1,$$

where k is a constant, does not have a unique solution. [2]

(b) Given that $k=-4$, show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

(c) Given instead that $k=-2$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [2]

(d) For the case where $k\neq-2$ and $k\neq-4$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [2]

34. [9231/w20/21/q7]

The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) State the eigenvalues of **P**. [1]

(b) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$. [4]

The 3×3 matrix **A** has distinct eigenvalues b, -1, 1 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

respectively.

(c) Find **A** in terms of b. [4]

35. [9231/w20/22/q9]

It is given that a is a positive constant.

(a) Show that the system of equations

$$\begin{aligned} ax + (2a+5)y + (a+1)z &= 1, \\ -4y \quad &= 2, \\ 3y - z &= 3, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}.$$

(b) Show that the eigenvalues of $\mathbf{A}$ are a, -1 and -4. [2]

(c) Find a matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}\begin{pmatrix} a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}\mathbf{P}^{-1}.$$ [5]

(d) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [6]

36. [9231/s19/11/q9]

It is given that **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ.

(i) Show that **e** is an eigenvector of $\mathbf{A}^2$, with corresponding eigenvalue λ^2. [2]

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + n\mathbf{I})^2,$$

where **I** is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find, in terms of n, a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [8]

37. [9231/s19/13/q11e]

Answer only **one** of the following two alternatives.

EITHER

A 3×3 matrix **A** has distinct eigenvalues 2, 1, 3, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

respectively, where b is a positive constant.

(i) Find **A** in terms of b. [9]

(ii) Find $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$. [2]

(iii) It is given that

$$\mathbf{A}^n\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^n\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b^{-1} \end{pmatrix}.$$

Find the values of n and b. [3]

38. [9231/w19/11/q8]

The matrix **M** is defined by

$$\mathbf{M} = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix},$$

where $m \neq 0,\ 1,\ 2$.

(i) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$. [7]

(ii) Find $\mathbf{M}^7\mathbf{P}$. [3]

39. [9231/w19/11/q10]

The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 \\ 1 & -2 & -2 \\ 2 & 3 & \theta \end{pmatrix}.$$

(i) (a) Find the rank of **A** when $\theta \neq -1$. [3]

(b) Find the rank of **A** when $\theta = -1$. [1]

Consider the system of equations

$$\begin{aligned} x + 5y + z &= -1, \\ x - 2y - 2z &= 0, \\ 2x + 3y + \theta z &= \theta. \end{aligned}$$

(ii) Solve the system of equations when $\theta \neq -1$. [3]

(iii) Find the general solution when $\theta = -1$. [3]

(iv) Show that if $\theta = -1$ and $\phi \neq -1$ then $\mathbf{Ax} = \begin{pmatrix} -1 \\ 0 \\ \phi \end{pmatrix}$ has no solution. [2]

40. [9231/s18/11/q11o]

It is given that **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ.

(i) Write down another eigenvector of **A** corresponding to λ. [1]

(ii) Write down an eigenvector and corresponding eigenvalue of $\mathbf{A}^n$, where n is a positive integer. [2]

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$.

(iii) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^n = \mathbf{PDP}^{-1}$. [7]

(iv) Determine the set of values of the real constant k such that

$$\sum_{n=1}^{\infty} k^n(\mathbf{A}^n - k\mathbf{A}^{n+1}) = k\mathbf{A}.$$ [4]

41. [9231/s18/13/q5]

It is given that **e** is an eigenvector of the matrix **A** with corresponding eigenvalue λ.

(i) Show that **e** is an eigenvector of $\mathbf{A}^3$ and state the corresponding eigenvalue. [3]

It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

(ii) Find a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{A}^3 + \mathbf{I} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where **I** is the 2×2 identity matrix. [5]

42. [9231/w18/11/q5]

It is given that λ is an eigenvalue of the matrix **A** with **e** as a corresponding eigenvector, and μ is an eigenvalue of the matrix **B** for which **e** is also a corresponding eigenvector.

(i) Show that $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A} + \mathbf{B}$ with **e** as a corresponding eigenvector. [2]

The matrix **A**, given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as eigenvectors.

(ii) Find the corresponding eigenvalues. [3]

The matrix **B** has eigenvalues 4, 5 and 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ respectively.

(iii) Find a matrix **P** and a diagonal matrix **D** such that $(\mathbf{A} + \mathbf{B})^3 = \mathbf{PDP}^{-1}$. [3]

43. [9231/w18/12/q2]

It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. [1]

(ii) Write down the negative eigenvalue of **A** and find a corresponding eigenvector. [3]

(iii) Find an eigenvalue and a corresponding eigenvector of the matrix $\mathbf{A} + \mathbf{A}^6$. [2]

44. [9231/s17/11/q4]

(i) Find the value of k for which the set of linear equations

$$\begin{aligned} x + 3y + kz &= 4, \\ 4x - 2y - 10z &= -5, \\ x + y + 2z &= 1, \end{aligned}$$

has no unique solution. [3]

(ii) For this value of k, find the set of possible solutions, giving your answer in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{a} + t\mathbf{b},$$

where **a** and **b** are vectors and t is a scalar. [3]

45. [9231/s17/11/q5]

The matrix **A**, given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ 6 & 4 & -6 \\ 6 & 5 & -7 \end{pmatrix},$$

has eigenvalues 1, −1 and −2.

(i) Find a set of corresponding eigenvectors. [4]

(ii) The matrix **B** is given by $\mathbf{B} = \mathbf{A} - 2\mathbf{I}$, where **I** is the 3×3 identity matrix. Write down the eigenvalues of **B**, and state a set of corresponding eigenvectors. [2]

46. [9231/s17/13/q10]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}.$$

(i) Given that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of **A**, find the corresponding eigenvalue. [2]

(ii) Given also that -1 is an eigenvalue of **A**, find a corresponding eigenvector. [2]

(iii) It is given that the determinant of **A** is equal to the product of the eigenvalues of **A**. Use this result to find the third eigenvalue of **A**, and find also a corresponding eigenvector. [3]

(iv) Write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where **D** is a diagonal matrix, and hence find the matrix $\mathbf{A}^n$ in terms of n, where n is a positive integer. [6]

47. [9231/w17/11/q11e]

(i) The vector **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ, and is also an eigenvector of the matrix **B**, with corresponding eigenvalue μ. Show that **e** is an eigenvector of the matrix **AB** with corresponding eigenvalue $\lambda\mu$. [3]

(ii) Find the eigenvalues and corresponding eigenvectors of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix}.$$ [6]

(iii) The matrix **B**, where

$$\mathbf{B} = \begin{pmatrix} 3 & 6 & 1 \\ 1 & -2 & -1 \\ 6 & 6 & -2 \end{pmatrix},$$

has eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find the eigenvalues of the matrix **AB**, and state corresponding eigenvectors. [4]

48. [9231/s16/11/q10]

Write down the eigenvalues of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

and find corresponding eigenvectors. [4]

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, and hence find the matrix $\mathbf{A}^n$, where n is a positive integer. [8]

49. [9231/s16/13/q3]

Find the two values of the constant k for which the equations

$$\begin{aligned} kx + y + z &= 2, \\ x + ky + z &= -1, \\ x + y + kz &= -1, \end{aligned}$$

have no unique solution. [4]

Show that, for one of these values of k, the equations have no solution, and solve the equations for the other value of k. [3]

50. [9231/s16/13/q11e]

It is given that 1 and 4 are eigenvalues of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix}.$$

Find eigenvectors corresponding to each of these eigenvalues. [3]

Given further that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of **A**, find the corresponding eigenvalue. [2]

Write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where **D** is a diagonal matrix, and find $\mathbf{P}^{-1}$. [5]

Write down a matrix **C** such that $\mathbf{C}^2 = \mathbf{D}$, and deduce a matrix **B** such that $\mathbf{B}^2 = \mathbf{A}$. [4]

51. [9231/w16/11/q3]

Find a matrix $\mathbf{A}$ whose eigenvalues are -1, 1, 2 and for which corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

respectively. [7]

52. [9231/s15/11/q2]

Find the value of the constant k for which the system of equations

$$\begin{aligned} 2x - 3y + 4z &= 1, \\ 3x - y &= 2, \\ x + 2y + kz &= 1, \end{aligned}$$

does not have a unique solution. [2]

For this value of k, solve the system of equations. [4]

53. [9231/s15/11/q10]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}.$$

The matrix **A** has an eigenvector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find the corresponding eigenvalue. [2]

The matrix **A** also has eigenvalues 4 and 6. Find corresponding eigenvectors. [3]

Hence find a matrix **P** such that $\mathbf{A} = \mathbf{PDP}^{-1}$, where **D** is a diagonal matrix which is to be determined. [2]

The matrix **B** is such that $\mathbf{B} = \mathbf{QAQ}^{-1}$, where

$$\mathbf{Q} = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

By using the expression $\mathbf{PDP}^{-1}$ for **A**, find the set of eigenvalues and a corresponding set of eigenvectors for **B**. [5]

54. [9231/s15/13/q11o]

One of the eigenvalues of the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 2 \\ -4 & \alpha & 6 \\ 2 & 6 & -2 \end{pmatrix},$$

is −9. Find the value of α. [3]

Find

(i) the other two eigenvalues, λ_1 and λ_2, of **M**, where $\lambda_1 > \lambda_2$, [5]

(ii) corresponding eigenvectors for all three eigenvalues of **M**. [3]

It is given that $\mathbf{x} = \boldsymbol{a}\mathbf{e}_1 + \boldsymbol{b}\mathbf{e}_2$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of **M** corresponding to the eigenvalues λ_1 and λ_2 respectively, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are scalar constants. Show that $\mathbf{Mx} = \boldsymbol{p}\mathbf{e}_1 + \boldsymbol{q}\mathbf{e}_2$, expressing $\boldsymbol{p}$ and $\boldsymbol{q}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. [3]

55. [9231/w15/11/q6]

The matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix},$$

has eigenvalues 1 and 3. Find corresponding eigenvectors. [3]

It is given that $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of **A**. Find the corresponding eigenvalue. [2]

Find a diagonal matrix **D** and matrices **P** and $\mathbf{P}^{-1}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. [5]

Chapter 3

Differentiation

1. **[9231/s25/21/q6]**

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2 u = \operatorname{sech}^2 u.$$ [3]

(b) Show that $\frac{d}{dt}(\operatorname{sech}^{-1} t) = -\frac{1}{t\sqrt{1-t^2}}$. [4]

It is given that

$$x = \tanh^{-1} t \quad \text{and} \quad y = t \operatorname{sech}^{-1} t, \quad \text{for } 0 < t < 1.$$

(c) Show that $\frac{dy}{dx} = -\sqrt{1-t^2} + (1-t^2)\operatorname{sech}^{-1} t$. [4]

(d) Find $\frac{d^2y}{dx^2}$ in terms of t. [4]

2. [9231/s25/23/q1]

Find the Maclaurin's series for $e^{\left(\frac{1}{x+2}\right)}$ up to and including the term in x^2. [5]

3. [9231/s25/23/q3]

The curve C has equation

$$9y^2 - 3\sinh^{-1}(xy) = 1 - 3\ln 3.$$

(a) Show that, at the point $(4, \frac{1}{3})$ on C, $\frac{dy}{dx} = -\frac{1}{2}$. [4]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(4, \frac{1}{3})$. [5]

4. [9231/s25/24/q4]

A curve has parametric equations

$$x = t^3 - t^2 + t - 1 \qquad \text{and} \qquad y = te^t.$$

(a) Show that 1 is the only real value of t for which $x = 0$. [1]

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(b) Show that $\dfrac{dy}{dx} = \dfrac{(t+1)e^t}{3t^2 - 2t + 1}$. [3]

(c) Find the Maclaurin's series for y up to and including the term in x^2. [6]

5. [9231/w25/21/q2]

The curve C has parametric equations

$$x = \sinh t \quad \text{and} \quad y = t + \cosh t.$$

(a) Find $\frac{dy}{dx}$ in terms of t. [2]

(b) Show that $\frac{d^2y}{dx^2} = \frac{1 - \sinh t}{\cosh^3 t}$. [4]

(c) Find the Maclaurin's series for y in terms of x up to and including the term in x^2. [2]

6. [9231/w25/22/q3]

The variables x and y are such that $y = 2$ when $x = -1$ and

$$x^2y + (x+y)^3 = 3.$$

(a) Show that $\frac{dy}{dx} = \frac{1}{4}$ when $x = -1$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ when $x = -1$. [6]

7. [9231/w25/24/q7]

The curve C has parametric equations

$$x = 8\ln(\tan\tfrac{1}{2}t) - 3\cot t - 3t, \qquad y = 6\ln(\tan\tfrac{1}{2}t) + 4\cot t + 4t, \qquad \text{for } \tfrac{1}{6}\pi \leqslant t \leqslant \tfrac{1}{3}\pi.$$

(a) **(i)** Show that $\dfrac{dx}{dt} = 8\operatorname{cosec} t + 3\cot^2 t$. [1]

(ii) Find $\dfrac{dy}{dt}$. [1]

(iii) Find the exact value of the length of C. [6]

(b) Show that

$$\frac{d^2y}{dx^2} = \frac{a\cot t\operatorname{cosec} t(\operatorname{cosec}^2 t + 1)}{(b\operatorname{cosec} t + c\cot^2 t)^n},$$

where a, b, c and n are integers to be determined. [5]

8. [9231/s24/21/q2]

Find the Maclaurin's series for $e^{1+x^2} + e^{1-x}$ up to and including the term in x^2. [4]

9. [9231/s24/21/q3]

It is given that

$$x = \sin^{-1} t \quad \text{and} \quad y = t\cos^{-1} t, \quad \text{for } 0 \leqslant t < 1.$$

(a) Show that $\frac{dy}{dx} = -t + \sqrt{1-t^2}\cos^{-1} t$. [3]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t. [4]

10. [9231/s24/23/q2]

The curve C has parametric equations

$$x = \cosh t\,, \quad y = \sinh t\,, \quad \text{for } 0 < t \leqslant \tfrac{3}{5}.$$

The length of C is denoted by s.

(a) Show that $s = \displaystyle\int_0^{\frac{3}{5}} \sqrt{\cosh 2t}\,\mathrm{d}t$. [4]

(b) By finding the Maclaurin's series for $\sqrt{\cosh 2t}$ up to and including the term in t^2, deduce an approximation to s. [5]

11. [9231/s24/23/q3]

The curve C has equation

$$x^3 + 2xy + 8y^3 = -12.$$

(a) Show that, at the point $(-2,-1)$ on C, $\dfrac{dy}{dx} = -\dfrac{1}{2}$. [3]

(b) Find the value of $\dfrac{d^2y}{dx^2}$ at the point $(-2,-1)$. [5]

12. [9231/w24/21/q2]

It is given that

$$x = 1 + \frac{1}{t} \quad \text{and} \quad y = \cos^{-1} t \quad \text{for } 0 < t < 1.$$

(a) Show that $\frac{dy}{dx} = \frac{t^2}{\sqrt{1-t^2}}$. [2]

(b) Show that $\frac{d^2y}{dx^2} = -t^a\left(1-t^2\right)^b\left(2-t^2\right)$, where a and b are constants to be determined. [4]

13. [9231/w24/22/q2]

The curve C has equation

$$4y^2 + 4\ln(xy) = 1.$$

(a) Show that, at the point $\left(2, \frac{1}{2}\right)$ on C, $\frac{dy}{dx} = -\frac{1}{6}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $\left(2, \frac{1}{2}\right)$. [4]

14. [9231/s23/21/q8]

(a) Starting from the definitions of sech and tanh in terms of exponentials, prove that

$$1 - \operatorname{sech}^2 t = \tanh^2 t.$$ [3]

The curve C has parametric equations

$$x = \tfrac{1}{2}\tanh^2 t + \ln \operatorname{sech} t, \qquad y = 1 + \tanh^4 t, \qquad \text{for } t > 0.$$

(b) Show that $\frac{dy}{dx} = -4\operatorname{sech}^2 t$. [5]

(c) Find the coordinates of the point on C with $\frac{d^2y}{dx^2} = -\frac{9}{2}$, giving your answer in the form $(a + \ln b, c)$ where a, b and c are rational numbers. [6]

15. [9231/s23/23/q1]

(a) Find the Maclaurin series for $\sin^{-1}x$ up to and including the term in x^3. [5]

(b) Deduce an approximation to $\int_0^{\frac{1}{5}} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u$, giving your answer as a fraction. [1]

16. [9231/s23/23/q4]

The curve C has equation

$$4y^3 + (x+y)^6 = 109.$$

(a) Show that, at the point $(-4, 3)$ on C, $\frac{dy}{dx} = \frac{1}{17}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-4, 3)$. [5]

17. [9231/w23/21/q3]

Find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\frac{1}{2}e^x\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [6]

18. [9231/w23/22/q1]

Find the Maclaurin's series for $\ln(x+2)+\ln(x^2+5)$ up to and including the term in x^2. [5]

19. [9231/w23/22/q2]

It is given that

$$x = 1 + \frac{1}{t} \qquad \text{and} \qquad y = te^t.$$

(a) Show that $\frac{dy}{dx} = -e^t(t^3 + t^2)$. [3]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t. [4]

20. [9231/s22/21/q5]

The variables x and y are such that $y = 0$ when $x = 0$ and

$$(x+1)y + (x+y+1)^3 = 1.$$

(a) Show that $\frac{dy}{dx} = -\frac{3}{4}$ when $x = 0$. [3]

(b) Find the Maclaurin's series for y up to and including the term in x^2. [7]

21. [9231/s22/23/q2.a]

(a) Find the coefficient of x^2 in the Maclaurin's series for $-\ln\cos x$. [4]

(b) Find the length of the arc of the curve with equation $y = -\ln\cos x$ from the point where $x = 0$ to the point where $x = \frac{1}{4}\pi$. [4]

22. [9231/s22/23/q4]

It is given that

$$x = -t + \tan^{-1} t \qquad \text{and} \qquad y = t + \sinh^{-1} t.$$

(a) Show that $\dfrac{dy}{dx} = -\dfrac{t^2 + 1 + \sqrt{t^2+1}}{t^2}$. [4]

(b) Find the value of $\dfrac{d^2y}{dx^2}$ when $t = \frac{3}{4}$. [5]

23. [9231/w22/21/q1]

Find the Maclaurin's series for $\ln(1+e^x)$ up to and including the term in x^2. [5]

24. [9231/w22/21/q4.b]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$ [3]

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}(\sinh x)\right) = \operatorname{sech} x$. [3]

(c) Sketch the graph of $y = \operatorname{sech} x$, stating the equation of the asymptote. [2]

(d) By considering a suitable set of n rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{n} \operatorname{sech} r < \tan^{-1}(\sinh n).$$ [3]

(e) Hence state an upper bound, in terms of π, for $\sum_{r=1}^{\infty} \operatorname{sech} r$. [1]

25. [9231/w22/22/q2]

A curve has equation

$$(x+1)y+y^2=2.$$

(a) Show that $\frac{dy}{dx}=-\frac{2}{3}$ at the point $(0,-2)$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(0,-2)$. [4]

26. [9231/w22/22/q3.b]

(a) A curve has equation $y = e^x + \frac{1}{4}e^{-x}$, for $0 \leqslant x \leqslant 1$. Find, in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis. [6]

(b) Using standard results from the list of formulae (MF19), or otherwise, find the Maclaurin's series for $e^x + \frac{1}{4}e^{-x}$ up to and including the term in x^2. [2]

27. [9231/s21/21/q7]

(a) It is given that $y = \operatorname{sech}^{-1}\left(x+\frac{1}{2}\right)$.

Express $\cosh y$ in terms of x and hence show that $\sinh y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(x+\frac{1}{2}\right)^2}$. [3]

(b) Find the first three terms in the Maclaurin's series for $\operatorname{sech}^{-1}\left(x+\frac{1}{2}\right)$ in the form

$$\ln a + bx + cx^2,$$

where a, b and c are constants to be determined. [7]

28. [9231/s21/21/q8.a]

The curve C has parametric equations

$$x = 2\cosh t,\quad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t,\quad \text{for } 0 \leqslant t \leqslant 1.$$

(a) Find $\dfrac{dx}{dt}$ and show that $\dfrac{dy}{dt} = 1 - \sinh^2 t$. [3]

The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) **(i)** Show that $A = \pi \displaystyle\int_0^1 \left(\tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt$. [4]

(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]

29. [9231/s21/23/q2]

Find the Maclaurin's series for $\ln \cosh x$ up to and including the term in x^4. [7]

30. [9231/w21/21/q1]

Find the Maclaurin's series for $e^x \tan x$ from first principles up to and including the term in x^2. [5]

31. [9231/w21/21/q3]

The curve C has equation

$$xy^3 - 4x^3y = 3.$$

(a) Show that, at the point $(-1, 1)$ on C, $\frac{dy}{dx} = 11$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-1, 1)$. [5]

32. [9231/w21/22/q1.ab]

It is given that $y = \sinh(x^2) + \cosh(x^2)$.

(a) Use standard results from the list of formulae (MF19) to find the Maclaurin's series for y in terms of x up to and including the term in x^4. [2]

(b) Deduce the value of $\dfrac{d^4y}{dx^4}$ when $x = 0$. [1]

(c) Use your answer to part **(a)** to find an approximation to $\int_0^{\frac{1}{2}} y \, dx$, giving your answer as a rational fraction in its lowest terms. [2]

33. [9231/w21/22/q5.b]

The curve C has parametric equations

$$x = 3t + 2t^{-1} + at^3, \quad y = 4t - \tfrac{3}{2}t^{-1} + bt^3, \quad \text{for } 1 \leqslant t \leqslant 2,$$

where a and b are constants.

(a) It is given that $a = \frac{2}{3}$ and $b = -\frac{1}{2}$.

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{25}{4}(t^2 + t^{-2})^2$ and find the exact length of C. [6]

(b) It is given instead that $a = b = 0$.

Find the value of $\frac{d^2y}{dx^2}$ when $t = 1$. [4]

34. [9231/s20/21/q2]

It is given that $y = 2^x$.

(a) By differentiating $\ln y$ with respect to x, show that $\frac{dy}{dx} = 2^x \ln 2$. [3]

(b) Write down $\frac{d^2y}{dx^2}$. [1]

(c) Hence find the first three terms in the Maclaurin's series for 2^x. [3]

35. [9231/s20/23/q5.b]

The curve C has parametric equations

$$x = \frac{1}{2}t^2 - \ln t, \qquad y = 2t + 1, \qquad \text{for } \frac{1}{2} \leqslant t \leqslant 2.$$

(a) Find the exact length of C. [5]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [4]

36. [9231/s20/23/q6.bc]

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1-\tanh^2\theta = \operatorname{sech}^2\theta.$$ [3]

The variables x and y are such that $\tanh y = \cos\left(x+\frac{1}{4}\pi\right)$, for $-\frac{1}{4}\pi < x < \frac{3}{4}\pi$.

(b) By differentiating the equation $\tanh y = \cos\left(x+\frac{1}{4}\pi\right)$ with respect to x, show that

$$\frac{dy}{dx} = -\operatorname{cosec}\left(x+\frac{1}{4}\pi\right).$$ [4]

(c) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\cos\left(x+\frac{1}{4}\pi\right)\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [5]

37. [9231/w20/21/q1.a]

(a) By differentiating e^{-x^2}, find the Maclaurin's series for e^{-x^2} up to and including the term in x^2. [5]

(b) Deduce an approximation to $\int_0^{\frac{1}{5}} e^{-x^2}\,dx$, giving your answer as a rational fraction in its lowest terms. [2]

38. [9231/w20/21/q5]

It is given that

$$x = \sinh^{-1} t, \quad y = \cos^{-1} t,$$

where $-1 < t < 1$.

(a) By differentiating $\cos y$ with respect to t, show that $\dfrac{dy}{dt} = -\dfrac{1}{\sqrt{1-t^2}}$. [4]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [5]

39. [9231/w20/21/q8.c]

(a) Sketch the graph of $y = \coth x$ for $x > 0$ and state the equations of the asymptotes. [2]

(b) Starting from the definitions of coth and cosech in terms of exponentials, prove that

$$\coth^2 x - \operatorname{cosech}^2 x = 1.$$ [3]

The curve C has equation $y = \ln \coth\left(\frac{1}{2}x\right)$ for $x > 0$.

(c) Show that $\frac{dy}{dx} = -\operatorname{cosech} x$. [3]

(d) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where a is a positive constant.

Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a. [7]

40. **[9231/w20/22/q1]**

Find the Maclaurin's series for $\tan\left(x+\frac{1}{4}\pi\right)$ up to and including the term in x^2. [5]

41. [9231/w20/22/q5]

The curve C has equation

$$y^2 + (xy+1)^2 = 5.$$

(a) Show that, at the point $(1,1)$ on C, $\frac{dy}{dx} = -\frac{2}{3}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(1,1)$. [5]

42. [9231/s19/11/q1]

A curve C has equation $\cos y = x$, for $-\pi < x < \pi$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -\cot y\left(\frac{dy}{dx}\right)^2. \qquad [4]$$

(ii) Hence find the exact value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\pi\right)$ on C. [2]

43. [9231/s19/13/q11o]

OR

The positive variables y and t are related by

$$y = a^t,$$

where a is a positive constant.

(i) (a) By differentiating $\ln y$ with respect to t, show that $\dfrac{dy}{dt} = a^t \ln a$. [3]

(b) Write down $\dfrac{d^2y}{dt^2}$. [1]

(ii) Determine the set of values of a for which the infinite series

$$y + \frac{dy}{dt} + \frac{d^2y}{dt^2} + \frac{d^3y}{dt^3} + \dots$$

is convergent. [3]

A curve has parametric equations

$$x = t^a, \quad y = a^t.$$

(iii) Find $\dfrac{d^2y}{dx^2}$ in terms of a and t, and show that, when $t = 2$,

$$\frac{d^2y}{dx^2} = 2^{1-2a}(1 - a + 2\ln a)\ln a.$$ [7]

44. [9231/s18/13/q1]

The variables x and y are such that $y = -1$ when $x = 0$ and

$$\left(x + \frac{dy}{dx}\right)^3 = y^2 + x.$$

(i) Find the value of $\frac{dy}{dx}$ when $x = 0$. [1]

(ii) Find also the value of $\frac{d^2y}{dx^2}$ when $x = 0$. [4]

45. [9231/w18/12/q6]

It is given that $y = e^x u$, where u is a function of x. The rth derivatives $\frac{d^r y}{dx^r}$ and $\frac{d^r u}{dx^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^x\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \ldots + \binom{n}{r}u^{(r)} + \ldots + \binom{n}{n}u^{(n)}\right).$$ [8]

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

46. [9231/w18/12/q11e]

The curve C is defined parametrically by

$$x = 18t - t^2 \quad \text{and} \quad y = 8t^{\frac{3}{2}},$$

where $0 < t \leqslant 4$.

(i) Show that at all points of C,

$$\frac{d^2y}{dx^2} = \frac{3(9+t)}{2t^{\frac{1}{2}}(9-t)^3}.$$ [4]

(ii) Show that the mean value of $\frac{d^2y}{dx^2}$ with respect to x over the interval $0 < x \leqslant 56$ is $\frac{3}{70}$. [4]

(iii) Find the area of the surface generated when C is rotated through 2π radians about the x-axis, showing full working. [6]

47. [9231/s17/11/q3]

A curve C has equation $\tan y = x$, for $x > 0$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -2x\left(\frac{dy}{dx}\right)^2.$$ [3]

(ii) Hence find the value of $\frac{d^2y}{dx^2}$ at the point $\left(1, \frac{1}{4}\pi\right)$ on C. [2]

48. [9231/s17/13/q4]

A curve C has equation $x^3 - 3xy + y^2 = 4$. Find the value of $\frac{d^2y}{dx^2}$ at the point (0, 2) of C. [7]

49. [9231/w17/11/q5]

The curve C has equation $2x^3 + 3x^2y - 3y^3 - 16 = 0$.

(i) Find the coordinates of the point A on C at which $\frac{dy}{dx} = 0$ and $x \neq 0$. [5]

(ii) Find the value of $\frac{d^2y}{dx^2}$ at A. [3]

50. [9231/w16/11/q8]

A curve C has equation $x^2 + 4xy - y^2 + 20 = 0$. Show that, at stationary points on C, $x = -2y$. [3]

Find the coordinates of the stationary points on C, and determine their nature by considering the value of $\frac{d^2y}{dx^2}$ at the stationary points. [8]

51. [9231/s15/11/q6]

A curve has equation $x^2 - 6xy + 25y^2 = 16$. Show that $\frac{dy}{dx} = 0$ at the point $(3, 1)$. [4]

By finding the value of $\frac{d^2y}{dx^2}$ at the point $(3, 1)$, determine the nature of this turning point. [5]

52. [9231/s15/13/q7]

The curve C has equation $x^2 + 2xy - 4y^2 + 20 = 0$. Show that if the tangent to C at the point (x, y) is parallel to the x-axis then $x + y = 0$. [3]

Hence find the coordinates of the stationary points on C, and determine their nature. [7]

53. [9231/w15/11/q1]

The curve C is defined parametrically by

$$x = 2\cos^3 t \quad \text{and} \quad y = 2\sin^3 t, \quad \text{for } 0 < t < \tfrac{1}{2}\pi.$$

Show that, at the point with parameter t,

$$\frac{d^2y}{dx^2} = \tfrac{1}{6}\sec^4 t \operatorname{cosec} t.$$ [4]

Chapter 4

Integration

1. [9231/s25/23/q2]

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$\tanh^2 t + \operatorname{sech}^2 t = 1.$$ [3]

(b) The curve C has parametric equations

$$x = \ln(\cosh t), \quad y = \tan^{-1}(\sinh t), \quad \text{for } 0 \leqslant t \leqslant 1.$$

Find the length of C. [5]

2. [9231/s25/24/q2]

Find the exact value of $\int_{1}^{\frac{5}{2}} \frac{1}{\sqrt{x^2-2x+5}}\,dx$, giving your answer in logarithmic form. [6]

3. [9231/s25/24/q8]

The curve C has equation $y = \tanh x$ for $x \geqslant 0$.

(a) Sketch C and state the equation of the asymptote. [2]

(b) By considering a suitable set of N rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{N} \tanh r > \ln(\cosh N). \qquad [3]$$

(c) The arc of C joining the point where $x = 0$ to the point where $x = \frac{1}{2}\ln 3$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Use the substitution $u = \sqrt{1 + \operatorname{sech}^4 x}$ to show that

$$S = \pi \int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\,du. \qquad [7]$$

(ii) Find the exact value of $\pi \int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\,du$. You need not simplify your answer. [3]

4. [9231/w25/22/q1]

The curve C has polar equation $r = e^{\frac{4}{3}\theta}$ for $0 \leqslant \theta \leqslant \pi$.

Find the length of C. Give your answer in an exact form. [4]

5. [9231/w25/24/q7]

The curve C has parametric equations

$$x = 8\ln(\tan\tfrac{1}{2}t) - 3\cot t - 3t, \quad y = 6\ln(\tan\tfrac{1}{2}t) + 4\cot t + 4t, \quad \text{for } \tfrac{1}{6}\pi \leqslant t \leqslant \tfrac{1}{3}\pi.$$

(a) **(i)** Show that $\dfrac{dx}{dt} = 8\,\text{cosec}\,t + 3\cot^2 t$. [1]

(ii) Find $\dfrac{dy}{dt}$. [1]

(iii) Find the exact value of the length of C. [6]

(b) Show that

$$\frac{d^2y}{dx^2} = \frac{a\cot t\,\text{cosec}\,t(\text{cosec}^2 t + 1)}{(b\,\text{cosec}\,t + c\cot^2 t)^n},$$

where a, b, c and n are integers to be determined. [5]

6. [9231/s24/23/q1]

Find the exact value of $\int_{2}^{\frac{7}{2}} \frac{1}{\sqrt{4x - x^2 - 1}} \, dx$. [5]

7. [9231/s24/23/q2]

The curve C has parametric equations

$$x = \cosh t\,, \qquad y = \sinh t\,, \qquad \text{for } 0 < t \leqslant \tfrac{3}{5}.$$

The length of C is denoted by s.

(a) Show that $s = \int_0^{\frac{3}{5}} \sqrt{\cosh 2t}\,\mathrm{d}t$. [4]

(b) By finding the Maclaurin's series for $\sqrt{\cosh 2t}$ up to and including the term in t^2, deduce an approximation to s. [5]

8. [9231/w24/21/q3]

A curve has equation $y = e^x$ for $\ln\frac{4}{3} \leqslant x \leqslant \ln\frac{12}{5}$. The area of the surface generated when the curve is rotated through 2π radians about the x-axis is denoted by A.

(a) Use the substitution $u = e^x$ to show that

$$A = 2\pi\int_{\frac{4}{3}}^{\frac{12}{5}} \sqrt{1+u^2}\,du.$$ [2]

(b) Use the substitution $u = \sinh v$ to show that

$$A = \pi\left(\frac{904}{225} + \ln\frac{5}{3}\right).$$ [6]

9. [9231/w24/22/q1]

Find the value of $\int_6^7 \frac{1}{\sqrt{(x-5)^2-1}}\,\mathrm{d}x$, giving your answer in the form $\ln(a+\sqrt{b})$, where a and b are integers to be determined. [4]

10. [9231/w24/22/q3]

The curve C has parametric equations

$$x = \frac{1}{2}e^{2t} - \frac{1}{3}t^3 - \frac{1}{2}, \qquad y = 2e^t(t-1), \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Find the exact length of C. [7]

11. [9231/w23/21/q5]

The curve C has parametric equations

$$x = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}, \qquad y = 2t + 5, \qquad \text{for } 0 < t \leqslant 3.$$

(a) Find the exact length of C. [5]

(b) Find the set of values of t for which $\dfrac{d^2y}{dx^2} > 0$. [5]

12. [9231/w23/22/q7]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\sinh^2 A = \cosh 2A - 1.$$ [3]

(b) A curve has equation $y = x^2$, for $0 \leqslant x \leqslant \frac{2}{3}$. The area of the surface generated when the curve is rotated through 2π radians about the x-axis is denoted by S.

Use the substitution $x = \frac{1}{2}\sinh u$ to show that $S = \frac{1}{32}\pi\left(\frac{820}{81} - \ln 3\right)$. [9]

13. [9231/s22/21/q1]

The curve C has polar equation $r = e^{\frac{3}{4}\theta}$ for $0 \leqslant \theta \leqslant \alpha$.

Given that the length of C is s, find α in terms of s. [5]

14. [9231/s22/23/q2]

(a) Find the coefficient of x^2 in the Maclaurin's series for $-\ln\cos x$. [4]

(b) Find the length of the arc of the curve with equation $y = -\ln\cos x$ from the point where $x = 0$ to the point where $x = \frac{1}{4}\pi$. [4]

15. [9231/w22/21/q3]

The curve C has parametric equations

$$x = \mathrm{e}^t - \tfrac{1}{3}t^3, \quad y = 4\mathrm{e}^{\frac{1}{2}t}(t-2), \qquad \text{for } 0 \leqslant t \leqslant 2.$$

Find, in terms of e, the length of C. [6]

16. [9231/w22/22/q3]

(a) A curve has equation $y = e^x + \frac{1}{4}e^{-x}$, for $0 \leqslant x \leqslant 1$. Find, in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis. [6]

(b) Using standard results from the list of formulae (MF19), or otherwise, find the Maclaurin's series for $e^x + \frac{1}{4}e^{-x}$ up to and including the term in x^2. [2]

17. [9231/s21/21/q8]

The curve C has parametric equations

$$x = 2\cosh t, \quad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t, \quad \text{for } 0 \leqslant t \leqslant 1.$$

(a) Find $\dfrac{dx}{dt}$ and show that $\dfrac{dy}{dt} = 1 - \sinh^2 t$. [3]

The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) (i) Show that $A = \pi \int_0^1 \left(\tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt$. [4]

(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]

18. [9231/w21/21/q8]

(a) Starting from the definition of cosh in terms of exponentials, prove that

$$2\cosh^2 A = \cosh 2A + 1. \qquad [3]$$

The curve C has parametric equations

$$x = 2\cosh 2t + 3t, \quad y = \tfrac{3}{2}\cosh 2t - 4t, \quad \text{for } -\tfrac{1}{2} \leqslant t \leqslant \tfrac{1}{2}.$$

The area of the surface generated when C is rotated through 2π radians about the y-axis is denoted by A.

(b) (i) Show that $A = 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)\cosh 2t \, \mathrm{d}t$. [4]

(ii) Hence find A in terms of π and e. [7]

19. [9231/w21/22/q5]

The curve C has parametric equations

$$x = 3t + 2t^{-1} + at^3, \quad y = 4t - \tfrac{3}{2}t^{-1} + bt^3, \quad \text{for } 1 \leqslant t \leqslant 2,$$

where a and b are constants.

(a) It is given that $a = \frac{2}{3}$ and $b = -\frac{1}{2}$.

Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{25}{4}(t^2 + t^{-2})^2$ and find the exact length of C. [6]

(b) It is given instead that $a = b = 0$.

Find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $t = 1$. [4]

20. [9231/s20/21/q5]

The curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$ intersect at the point where $x = a$.

(a) Find the exact value of a, giving your answer in logarithmic form. [4]

(b) Sketch C_1 and C_2 on the same diagram. [2]

(c) Find the exact value of the length of the arc of C_1 from $x = 0$ to $x = a$. [5]

21. [9231/s20/23/q5]

The curve C has parametric equations

$$x = \tfrac{1}{2}t^2 - \ln t, \qquad y = 2t + 1, \qquad \text{for } \tfrac{1}{2} \leqslant t \leqslant 2.$$

(a) Find the exact length of C. [5]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [4]

22. [9231/w20/21/q8]

(a) Sketch the graph of $y = \coth x$ for $x > 0$ and state the equations of the asymptotes. [2]

(b) Starting from the definitions of coth and cosech in terms of exponentials, prove that

$$\coth^2 x - \operatorname{cosech}^2 x = 1.$$ [3]

The curve C has equation $y = \ln\coth\left(\frac{1}{2}x\right)$ for $x > 0$.

(c) Show that $\frac{dy}{dx} = -\operatorname{cosech} x$. [3]

(d) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where a is a positive constant.

Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a. [7]

23. [9231/w20/22/q2]

A curve has equation $y = \cosh x$, for $0 \leqslant x \leqslant \frac{1}{2}$.

Find, in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis. [6]

24. [9231/s19/11/q5]

A curve C is defined parametrically by

$$x = \frac{2}{e^t + e^{-t}} \quad \text{and} \quad y = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

for $0 \leqslant t \leqslant 1$. The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by S.

(i) Show that $S = 4\pi \int_0^1 \frac{e^t - e^{-t}}{\left(e^t + e^{-t}\right)^2}\, dt$. [5]

(ii) Using the substitution $u = e^t + e^{-t}$, or otherwise, find S in terms of π and e. [3]

25. [9231/s18/11/q1]

The curve C is defined parametrically by

$$x = e^t - t, \quad y = 4e^{\frac{1}{2}t}.$$

Find the length of the arc of C from the point where $t = 0$ to the point where $t = 3$. [5]

26. [9231/w18/11/q4]

A curve is defined parametrically by

$$x = t - \tfrac{1}{2}\sin 2t \quad \text{and} \quad y = \sin^2 t.$$

The arc of the curve joining the point where $t = 0$ to the point where $t = \pi$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Show that

$$S = a\pi \int_0^{\pi} \sin^3 t \, dt,$$

where the constant a is to be found. [5]

(ii) Using the result $\sin 3t = 3\sin t - 4\sin^3 t$, find the exact value of S. [3]

27. [9231/s17/11/q12e]

The curve C has equation $y = \frac{1}{2}(e^x + e^{-x})$ for $0 \leqslant x \leqslant 4$.

(i) The region R is bounded by C, the x-axis, the y-axis and the line $x = 4$. Find, in terms of e, the coordinates of the centroid of the region R. [10]

(ii) Show that $\frac{ds}{dx} = \frac{1}{2}(e^x + e^{-x})$, where s denotes the arc length of C, and find the surface area generated when C is rotated through 2π radians about the x-axis. [4]

28. [9231/s17/13/q5]

A curve C has parametric equations

$$x = \frac{2}{5}t^{\frac{5}{2}} - 2t^{\frac{1}{2}}, \quad y = \frac{4}{3}t^{\frac{3}{2}}, \quad \text{for } 1 \leqslant t \leqslant 4.$$

(i) Find the exact value of the arc length of C. [5]

(ii) Find also the exact value of the surface area generated when C is rotated through 2π radians about the x-axis. [3]

29. [9231/s16/11/q11e]

A curve C has parametric equations

$$x = e^{2t} \cos 2t, \quad y = e^{2t} \sin 2t, \quad \text{for } -\tfrac{1}{2}\pi \leqslant t \leqslant \tfrac{1}{2}\pi.$$

Find the arc length of C. [6]

Find the area of the surface generated when C is rotated through 2π radians about the x-axis. [8]

30. [9231/s16/13/q4]

The curve C has equation $y = -\ln(1 - x^2)$ for $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$. Show that

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(\frac{1 + x^2}{1 - x^2}\right)^2. \qquad [2]$$

Show further that $\frac{1 + x^2}{1 - x^2}$ may be expressed in the form $\frac{P}{1 + x} + \frac{Q}{1 - x} + R$, where P, Q and R are constants to be determined. [2]

Find the exact arc length of C. [4]

31. [9231/w16/11/q11o]

A curve C has parametric equations

$$x = 1 - 3t^2, \quad y = t(1 - 3t^2), \quad \text{for } 0 \leqslant t \leqslant \frac{1}{\sqrt{3}}.$$

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 + 9t^2\right)^2$. [2]

Hence find

(i) the arc length of C, [2]

(ii) the surface area generated when C is rotated through 2π radians about the x-axis. [3]

Use the fact that $t = \frac{y}{x}$ to find a cartesian equation of C. Hence show that the polar equation of C is $r = \sec\theta(1 - 3\tan^2\theta)$, and state the domain of θ. [4]

Find the area of the region enclosed between C and the initial line. [3]

32. [9231/s15/11/q9]

The curve C has parametric equations

$$x = 4t + 2t^{\frac{3}{2}}, \quad y = 4t - 2t^{\frac{3}{2}}, \quad \text{for } 0 \leqslant t \leqslant 4.$$

Find the arc length of C, giving your answer correct to 3 significant figures. [6]

Find the mean value of y with respect to x over the interval $0 \leqslant x \leqslant 32$. [5]

33. [9231/s15/13/q2]

The curve $\boldsymbol{C}$ has polar equation $\boldsymbol{r} = \mathrm{e}^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of $\boldsymbol{C}$ is 2015. Find the value of α. [6]

34. [9231/w15/11/q11o]

The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = 4a^2\sin^2\left(\tfrac{1}{2}\theta\right),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]

Chapter 5

Reduction formulae

1. [9231/s25/21/q2]

Let $I_n = \int_0^1 (1-x)^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$, $I_n = -1 + n(n-1)I_{n-2}$. [4]

(b) Find the exact value of I_2. [3]

2. [9231/w25/21/q3]

The integral I_n is defined by $I_n = \int_0^1 \left(1+x^2\right)^n \mathrm{d}x$.

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^n\right)$, or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}.$$ [5]

(b) Find the exact value of I_{-2}. [4]

3. [9231/w25/24/q1]

It is given that $I_n = \int_0^1 x^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$,

$$I_n = \cosh 1 - n \sinh 1 + n(n-1)I_{n-2}.$$ [3]

(b) Find the exact value of I_2. [2]

4. [9231/s24/21/q4]

It is given that, for $n \geqslant 0$, $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \, dx$.

(a) Show that, for $n \geqslant 2$,

$$(n-1)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right) + (n-2)I_{n-2}.$$ [5]

[You may use the result that $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.]

(b) Find the value of I_4. [3]

5. [9231/s23/21/q4]

The integral I_n is defined by $I_n = \int_0^1 (1+x^5)^n \, dx$.

(a) By considering $\frac{d}{dx}\left(x(1+x^5)^n\right)$, or otherwise, show that

$$(5n+1)I_n = 2^n + 5nI_{n-1}.$$ [5]

(b) Find the exact value of I_3. [4]

6. [9231/s23/23/q7]

The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}n} \mathrm{d}x$.

(a) Find the exact value of I_{-1} giving your answer in the form $\ln a$, where a is an integer to be determined. [2]

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^{\frac{1}{2}n}\right)$, or otherwise, show that

$$(n+1)I_n = nI_{n-2} + \tfrac{4}{3}\left(\tfrac{5}{3}\right)^n.$$ [5]

(c) A curve has equation $y = x^2$, for $0 \leqslant x \leqslant \frac{2}{3}$. The arc length of the curve is denoted by s.

Use the substitution $u = 2x$ to show that $s = \frac{1}{2}I_1$ and find the exact value of s. [4]

7. [9231/s22/23/q8]

(a) Find $\int \sin\theta \cos^n\theta \, d\theta$, where $n \neq -1$. [2]

Let $I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m\theta \cos^n\theta \, d\theta$.

(b) Show that, for $m \geqslant 2$ and $n \geqslant 0$,

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$ [5]

(c) By considering the binomial expansion of $\left(z + \frac{1}{z}\right)^5$, where $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that

$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta,$$

where a, b and c are constants to be determined. [5]

(d) Using the results given in parts **(b)** and **(c)**, find the exact value of $I_{2,5}$. [4]

8. [9231/s21/23/q7]

The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{3}{2}} \left(4+x^2\right)^{-\frac{1}{2}n} \mathrm{d}x$.

(a) Find the exact value of I_1, expressing your answer in logarithmic form. [3]

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4+x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$4nI_{n+2} = \tfrac{3}{2}\left(\tfrac{2}{5}\right)^n + (n-1)I_n.$$ [5]

(c) Find the value of I_5. [3]

9. [9231/w21/22/q8]

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1-\tanh^2 x = \operatorname{sech}^2 x.$$ [3]

(b) Using the substitution $u = \tanh x$, or otherwise, find $\int \operatorname{sech}^2 x \tanh^2 x \, dx$. [2]

It is given that, for $n \geqslant 0$, $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$.

(c) Show that, for $n \geqslant 2$,

$$(n+1)I_n = \left(\tfrac{4}{5}\right)^3 \left(\tfrac{3}{5}\right)^{n-2} + (n-2)I_{n-2}.$$ [5]

[You may use the result that $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.]

(d) Find the value of I_4. [3]

10. [9231/s20/21/q6]

The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{1}{2}} \left(1-x^2\right)^{-\frac{1}{2}n} \, \mathrm{d}x$.

(a) Find the exact value of I_1. [2]

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1-x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n.$$ [5]

(c) Find the exact value of I_5 giving the answer in the form $k\sqrt{3}$, where k is a rational number to be determined. [3]

11. [9231/s19/11/q4]

It is given that, for $n \geqslant 0$,

$$I_n = \int_0^1 x^n e^{x^3}\, dx.$$

(i) Show that $I_2 = \frac{1}{3}(e - 1)$. [2]

(ii) Show that, for $n \geqslant 3$,

$$3I_n = e - (n-2)I_{n-3}.$$ [3]

(iii) Hence find the exact value of I_8. [3]

12. [9231/s19/13/q10]

Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, dx$, where $n \geqslant 0$.

(i) By considering $\frac{d}{dx}(\cot^{n+1} x)$, or otherwise, show that

$$I_{n+2} = \frac{1}{n+1} - I_n.$$

[5]

The curve C has equation $y = \cot x$, for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(ii) Find, in an exact form, the y-coordinate of the centroid of the region enclosed by C, the line $x = \frac{1}{4}\pi$ and the x-axis. [6]

13. [9231/w19/11/q3]

The integral I_n, where n is a positive integer, is defined by

$$I_n = \int_{\frac{1}{2}}^{1} x^{-n} \sin \pi x \, dx.$$

(i) Show that

$$n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n.$$ [5]

(ii) Find I_5 in terms of π and I_1. [2]

14. [9231/s18/11/q9]

(i) Using the substitution $u = \tan x$, or otherwise, find $\int \sec^2 x \tan^2 x \, dx$. [2]

It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \tan^2 x \, dx.$$

(ii) Using the result that $\frac{d}{dx}(\sec x) = \tan x \sec x$, show that, for $n \geqslant 2$,

$$(n+1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}.$$ [5]

(iii) Hence find the mean value of $\sec^4 x \tan^2 x$ with respect to x over the interval $0 \leqslant x \leqslant \frac{1}{4}\pi$, giving your answer in exact form. [3]

15. [9231/s18/13/q11e]

(i) Show that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^x \cos x \, dx = \frac{1}{2}\left(e^{\frac{1}{2}\pi} + e^{-\frac{1}{2}\pi}\right). \qquad [4]$$

(ii) It is given that, for $n \geqslant 0$,

$$I_n = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2x} \cos^n x \, dx.$$

Show that, for $n \geqslant 2$,

$$4I_n = n(n-1)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2x} \sin^2 x \cos^{n-2} x \, dx - nI_n,$$

and deduce the reduction formula

$$(n^2+4)I_n = n(n-1)I_{n-2}. \qquad [6]$$

(iii) Using the result in part **(i)** and the reduction formula in part **(ii)**, find the y-coordinate of the centroid of the region bounded by the x-axis and the arc of the curve $y = e^x \cos x$ from $x = -\frac{1}{2}\pi$ to $x = \frac{1}{2}\pi$. Give your answer correct to 3 significant figures. [4]

16. [9231/w18/11/q11o]

The curve C has equation

$$x^2 + 2xy = y^3 - 2.$$

(i) Show that $A\,(-1,\ 1)$ is the only point on C with x-coordinate equal to -1. [2]

For $n \geqslant 1$, let A_n denote the value of $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ at the point $A\,(-1,\ 1)$.

(ii) Show that $A_1 = 0$. [3]

(iii) Show that $A_2 = \frac{2}{5}$. [3]

Let $I_n = \displaystyle\int_{-1}^{0} x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\,\mathrm{d}x$.

(iv) Show that for $n \geqslant 2$,

$$I_n = (-1)^{n+1} A_{n-1} - nI_{n-1}.$$ [3]

(v) Deduce the value of I_3 in terms of I_1. [2]

17. [9231/w18/12/q11o]

Let $I_n = \int_1^{\sqrt{2}} (x^2 - 1)^n \, dx$.

(i) Show that, for $n \geqslant 1$,

$$(2n+1)I_n = \sqrt{2} - 2nI_{n-1}. \qquad [5]$$

(ii) Using the substitution $x = \sec\theta$, show that

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \sec\theta \, d\theta. \qquad [4]$$

(iii) Deduce the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sin^7\theta}{\cos^8\theta} \, d\theta. \qquad [5]$$

18. [9231/s17/11/q6]

Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$.

(i) Prove that, for $n \geqslant 2$,

$$I_n + n(n-1)I_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}.$$ [4]

(ii) Calculate the exact value of I_1 and deduce the exact value of I_3. [3]

19. [9231/s17/13/q6]

Let I_n denote $\int_0^2 (4 + x^2)^{-n}\,\mathrm{d}x$.

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(4 + x^2)^{-n}\right)$ and hence show that

$$8nI_{n+1} = (2n - 1)I_n + 2 \times 8^{-n}.$$ [5]

(ii) Use the result for integrating $\frac{1}{x^2 + a^2}$ with respect to x, in the List of Formulae (MF10), to find the value of I_1 and deduce that

$$I_3 = \tfrac{3}{1024}\pi + \tfrac{1}{128}.$$ [5]

20. [9231/w17/11/q8]

Let $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, dx$ for $n > 0$.

(i) Find the value of I_2. [2]

(ii) Show that, for $n > 2$,

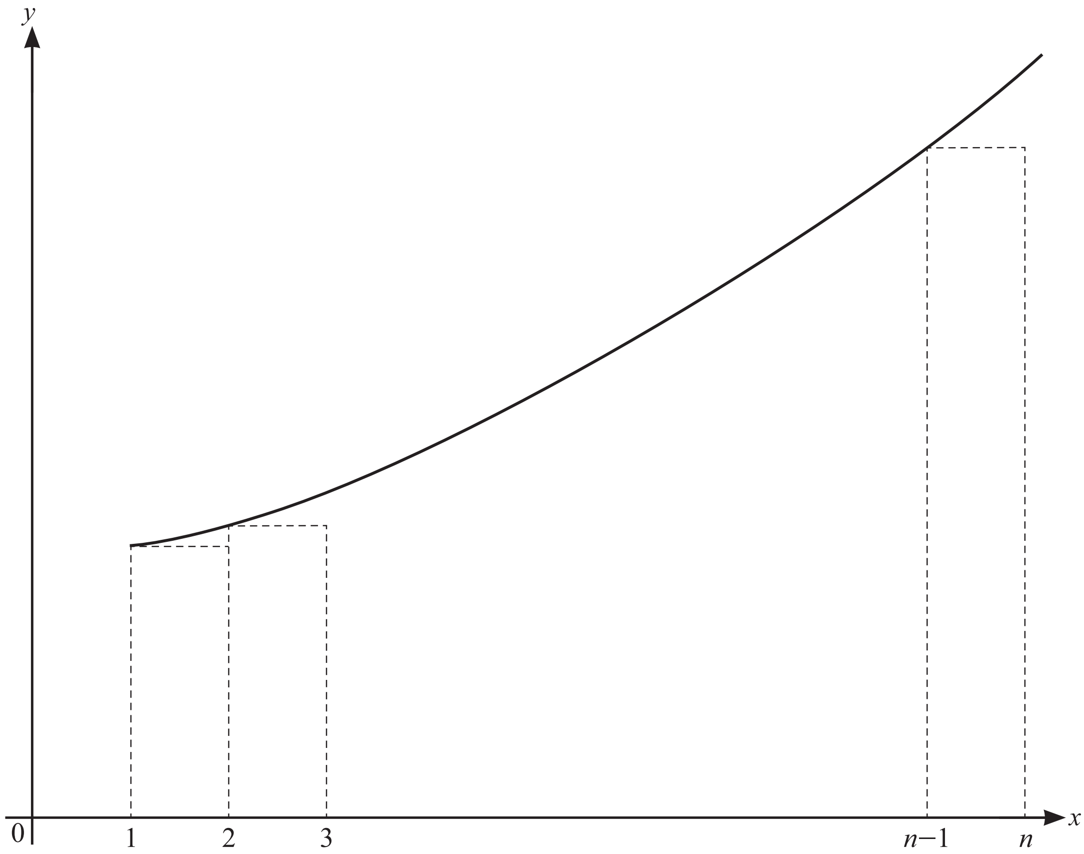
$$(n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2}.$$ [5]

(iii) The curve C has equation $y = \sec^3 x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The region R is bounded by C, the x-axis, the y-axis and the line $x = \frac{1}{4}\pi$. Find the volume of revolution generated when R is rotated through 2π radians about the x-axis. [4]

Chapter 6

Approximation of area or series summation

1. [9231/s25/21/q4]



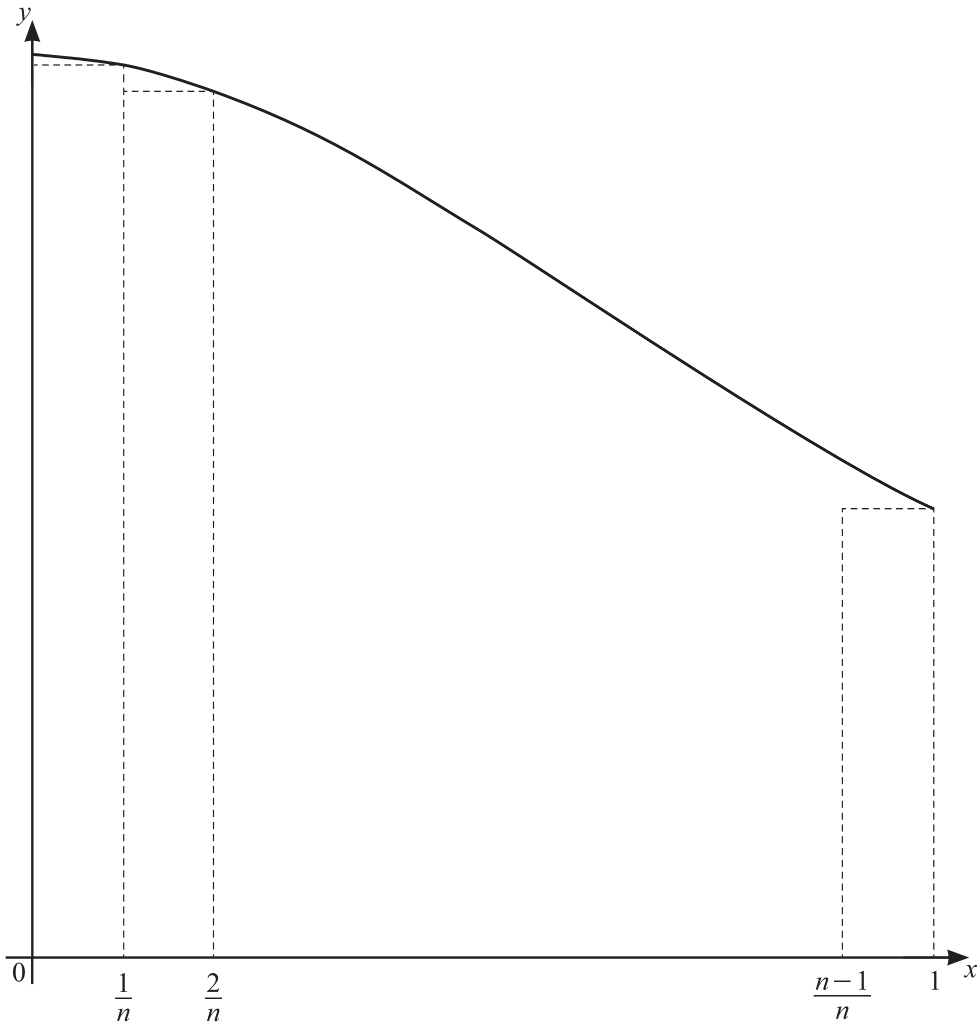
The diagram shows the curve with equation $y = \frac{1}{\sqrt{x}}e^{\sqrt{x}}$ for $x \geqslant 1$, together with a set of $n-1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}} < \left(2 + \frac{1}{\sqrt{n}}\right)e^{\sqrt{n}} - 2e.$$ [5]

(b) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}}$. [4]

2. [9231/s25/23/q6]



The diagram shows the curve with equation $y = \frac{1}{x^2+1}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{n}{n^2+r^2} < \frac{1}{4}\pi.$$ [5]

(b) Use a similar method to find a lower bound for $\sum_{r=1}^{n} \frac{n}{n^2+r^2}$. Give your answer in terms of n and π. [4]

(c) Deduce the exact value of $\lim_{n\to\infty} \sum^{n} \frac{n}{n^2+r^2}$. [1]

3. [9231/s25/24/q8]

The curve C has equation $y = \tanh x$ for $x \geqslant 0$.

(a) Sketch C and state the equation of the asymptote. [2]

(b) By considering a suitable set of N rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{N} \tanh r > \ln(\cosh N). \quad [3]$$

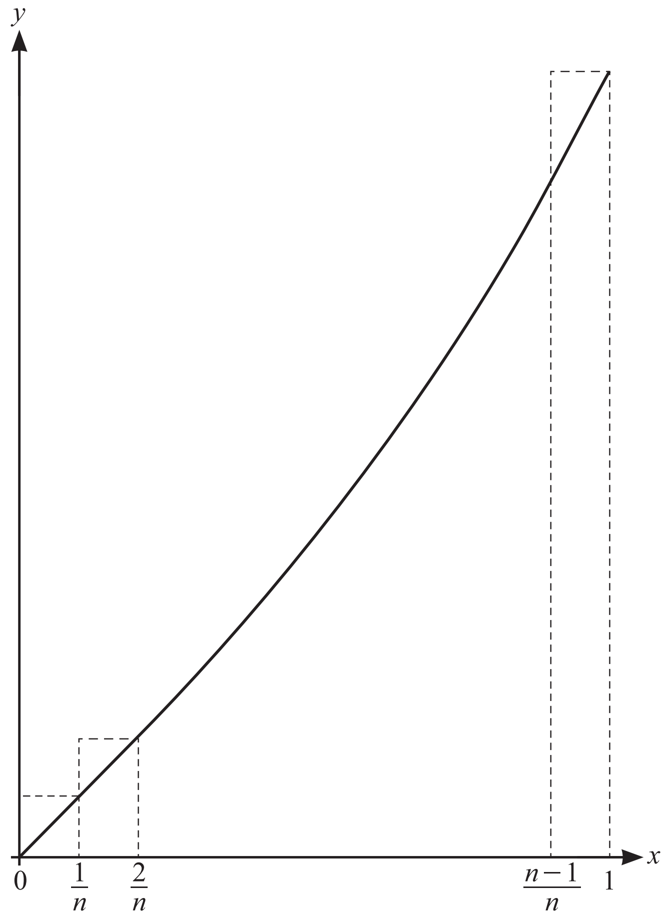
(c) The arc of C joining the point where $x = 0$ to the point where $x = \frac{1}{2}\ln 3$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Use the substitution $u = \sqrt{1 + \operatorname{sech}^4 x}$ to show that

$$S = \pi \int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\,\mathrm{d}u. \quad [7]$$

(ii) Find the exact value of $\pi \int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\,\mathrm{d}u$. You need not simplify your answer. [3]

4. [9231/w25/21/q5]



The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{3}x^3 + x\right)\mathrm{d}x < U_n$, where

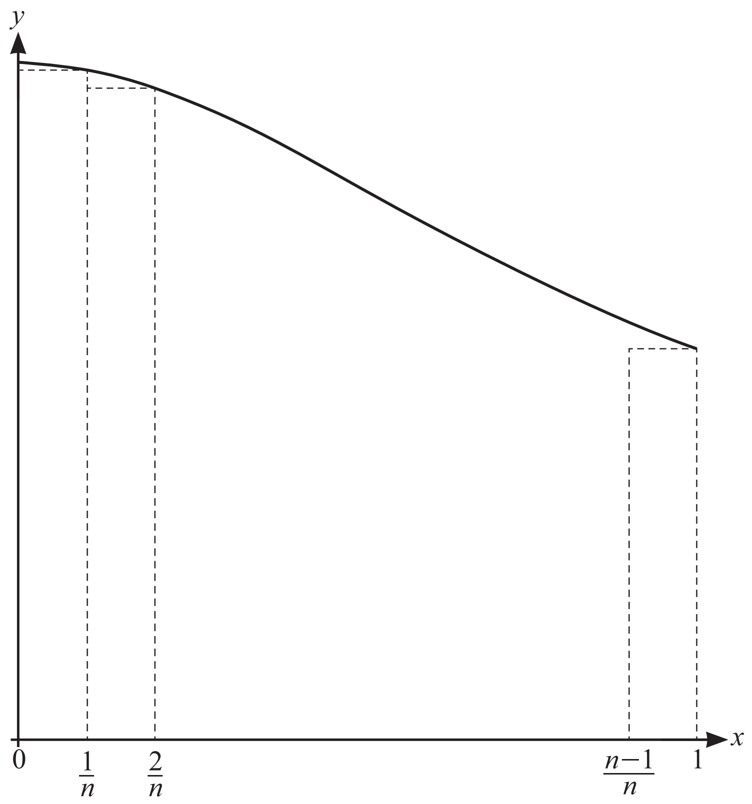
$$U_n = \frac{1}{12}\left(1 + \frac{1}{n}\right)\left(7 + \frac{1}{n}\right).$$ [5]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \left(\frac{1}{3}x^3 + x\right)\mathrm{d}x$. [4]

(c) Show that $\lim_{n \to \infty}(U_n - L_n) = 0$. [2]

5. [9231/w25/22/q6]

(a) Use the substitution $x = \frac{1}{2}\sqrt{2}\sinh u$ to find $\int \frac{1}{\sqrt{2x^2+1}}\,dx$. [3]



The diagram shows the curve with equation $y = \frac{1}{\sqrt{2x^2+1}}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

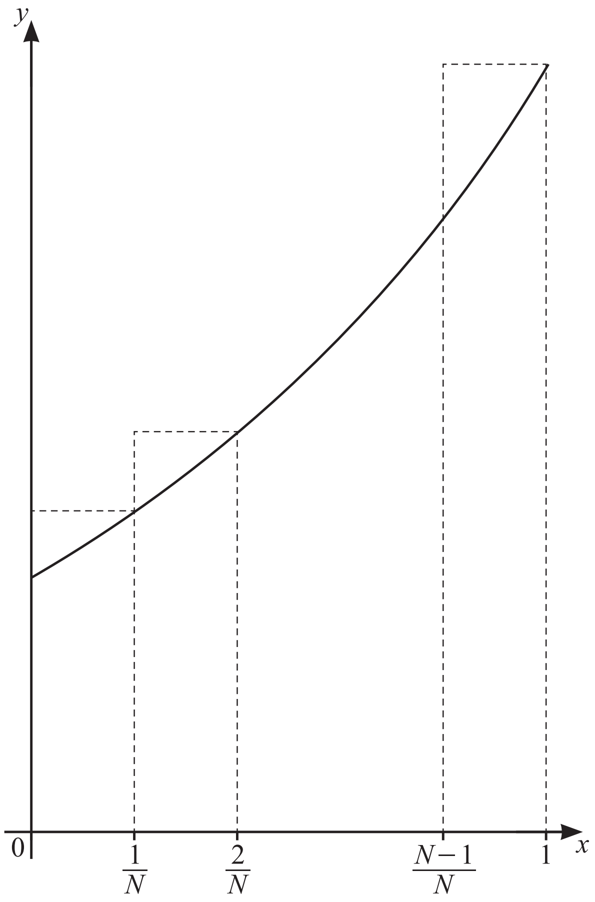
(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} < \frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right).$$ [5]

(c) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}}$. [4]

(d) Deduce the exact value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}}$. [1]

6. [9231/w25/24/q4]



The diagram shows the curve with equation $y = \frac{1}{2}(3^x)$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.

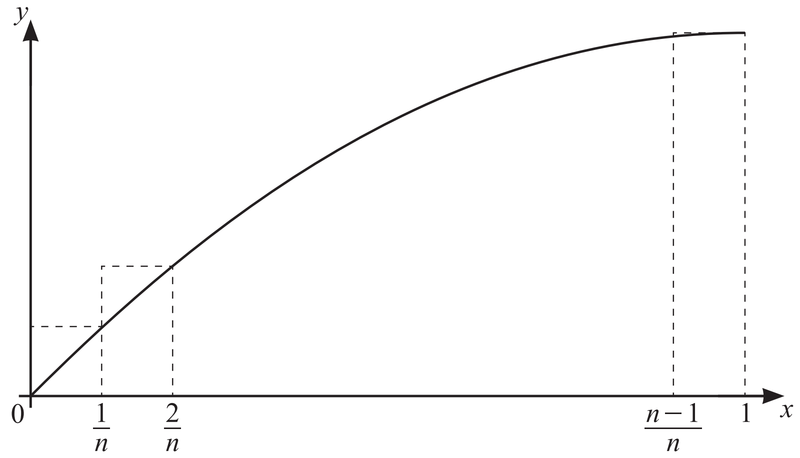
(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \frac{1}{2}(3^x)\,dx < U_N$, where

$$U_N = \frac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}} - 1\right)}.$$ [4]

(b) Use a similar method to find, in terms of N, a lower bound L_N for $\int_0^1 \frac{1}{2}(3^x)\,dx$. [4]

(c) By simplifying $U_N - L_N$, show that $\lim_{N\to\infty}\left(U_N - L_N\right) = 0$. [2]

7. [9231/s24/21/q5]



The diagram shows the curve with equation $y = 2x - x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

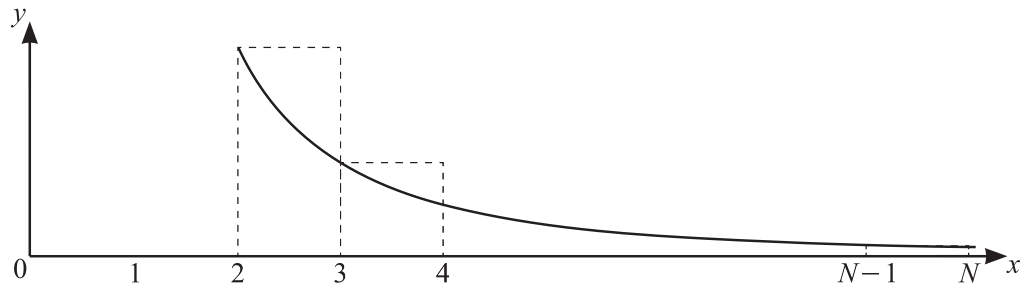
(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 (2x - x^2)\,dx < U_n$, where

$$U_n = \left(1 + \frac{1}{n}\right)\left(\frac{2}{3} - \frac{1}{6n}\right).$$ [5]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 (2x - x^2)\,dx$. [4]

(c) Show that $\lim_{n \to \infty}(U_n - L_n) = 0$. [2]

8. [9231/s24/23/q4]



The diagram shows the curve with equation $y = x^{-2}$ for $2 \leqslant x \leqslant N$ together with a set of $(N-2)$ rectangles of unit width.

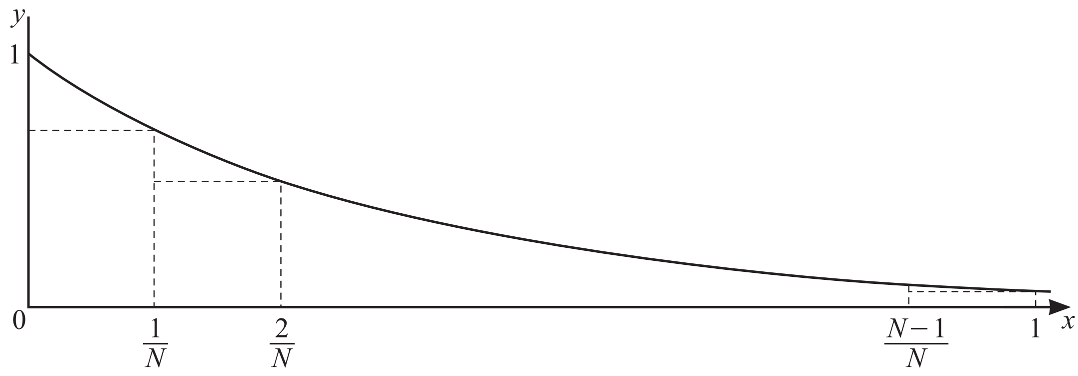
(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{1}{r^2} > \frac{3}{2} - \frac{1}{N} + \frac{1}{N^2}.$$ [5]

(b) Use a similar method to find, in terms of N, an upper bound for $\sum_{r=1}^{N} \frac{1}{r^2}$. [3]

(c) Deduce lower and upper bounds for $\sum_{r=1}^{\infty} \frac{1}{r^2}$. [2]

9. [9231/w24/21/q6]



The diagram shows the curve with equation $y = \left(\frac{1}{2}\right)^x$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{2}\right)^x \mathrm{d}x > L_N$, where

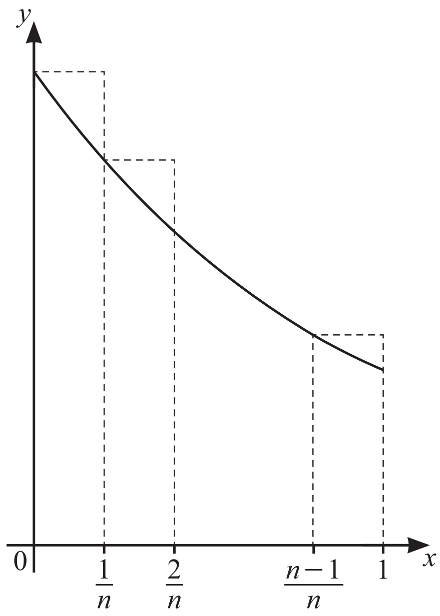
$$L_N = \frac{1}{2N\left(2^{\frac{1}{N}} - 1\right)}.$$ [4]

(b) Use a similar method to find, in terms of N, an upper bound U_N for $\int_0^1 \left(\frac{1}{2}\right)^x \mathrm{d}x$. [4]

(c) Find the least value of N such that $U_N - L_N \leqslant 10^{-3}$. [2]

(d) Given that $\int_0^1 \left(\frac{1}{2}\right)^x \mathrm{d}x = \frac{1}{2\ln 2}$, use the value of N found in part **(c)** to find upper and lower bounds for $\ln 2$. [4]

10. [9231/w24/22/q6]



The diagram shows the curve with equation $y = e^{1-x}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 e^{1-x}dx < U_n$, where

$$U_n = \frac{e-1}{n(1-e^{-\frac{1}{n}})}.$$ [4]

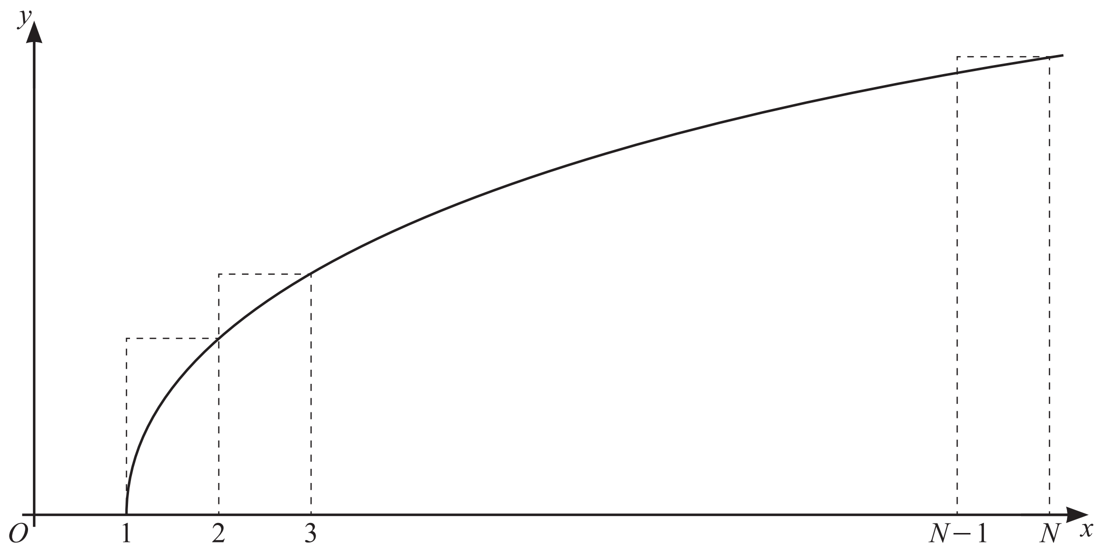
(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 e^{1-x}dx$. [4]

(c) Show that $\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

(d) Use the Maclaurin's series for e^x given in the list of formulae (MF19) to find the first three terms of the series expansion of $z(1-e^{-\frac{1}{z}})$, in ascending powers of $\frac{1}{z}$, and deduce the value of $\lim_{n\to\infty}(U_n)$. [3]

11. [9231/s23/21/q7]

(a) Use the substitution $u = x^2 - 1$ to find $\int \frac{x}{\sqrt{x^2-1}}\,dx$. [3]



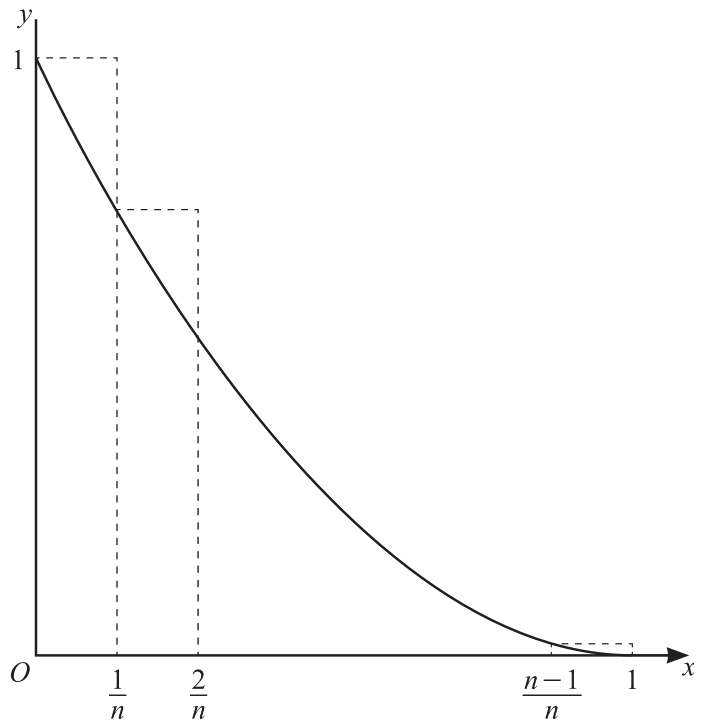
The diagram shows the curve with equation $y = \cosh^{-1} x$ together with a set of $(N-1)$ rectangles of unit width.

(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=2}^{N} \ln\left(r + \sqrt{r^2-1}\right) > N\ln\left(N + \sqrt{N^2-1}\right) - \sqrt{N^2-1}.$$ [5]

(c) Use a similar method to find, in terms of N, an upper bound for $\sum_{r=2}^{N} \ln\left(r + \sqrt{r^2-1}\right)$. [3]

12. [9231/s23/23/q6]



The diagram shows the curve with equation $y = (1-x)^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 (1-x)^2 \mathrm{d}x < U_n$, where

$$U_n = \frac{2n^2 + 3n + 1}{6n^2}.$$ [5]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 (1-x)^2 \mathrm{d}x$. [4]

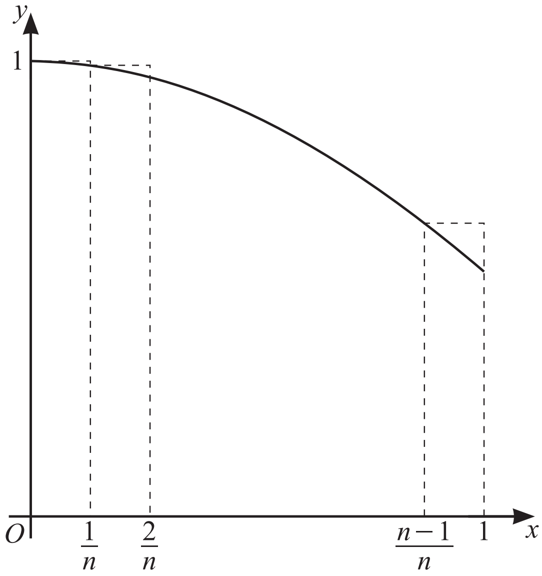
(c) Show that $\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

13. [9231/w23/21/q8]

(a) State the sum of the series $1+z+z^2+\ldots+z^{n-1}$, for $z \neq 1$. [1]

(b) By letting $z = \cos\theta + i\sin\theta$, where $\cos\theta \neq 1$, show that

$$1+\cos\theta+\cos 2\theta+\ldots+\cos(n-1)\theta = \frac{1}{2}\left(1-\cos n\theta+\frac{\sin n\theta\sin\theta}{1-\cos\theta}\right).$$ [7]



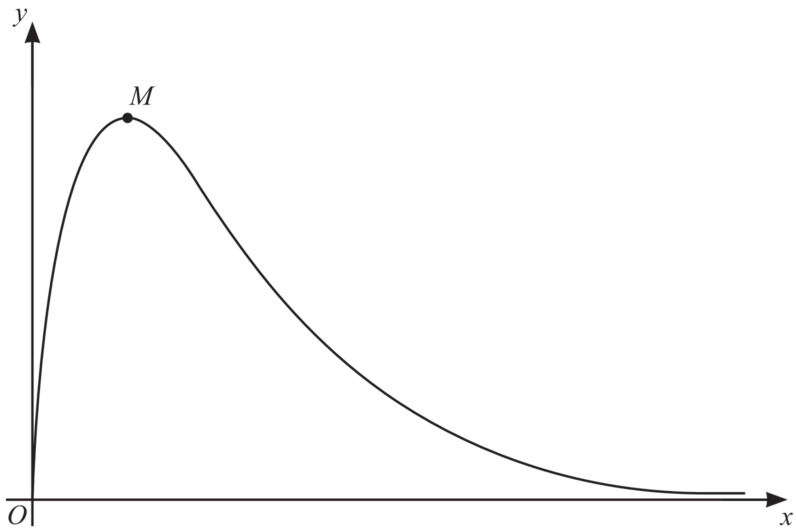
The diagram shows the curve with equation $y = \cos x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(c) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 \cos x\,dx < \frac{1}{2n}\left(1-\cos 1+\frac{\sin 1\sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right).$$ [4]

(d) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 \cos x\,dx$. [3]

14. [9231/w23/22/q5]



The diagram shows part of the curve $y = x\,\text{sech}^2 x$ and its maximum point M.

(a) Show that, at M,

$$2x\tanh x - 1 = 0$$

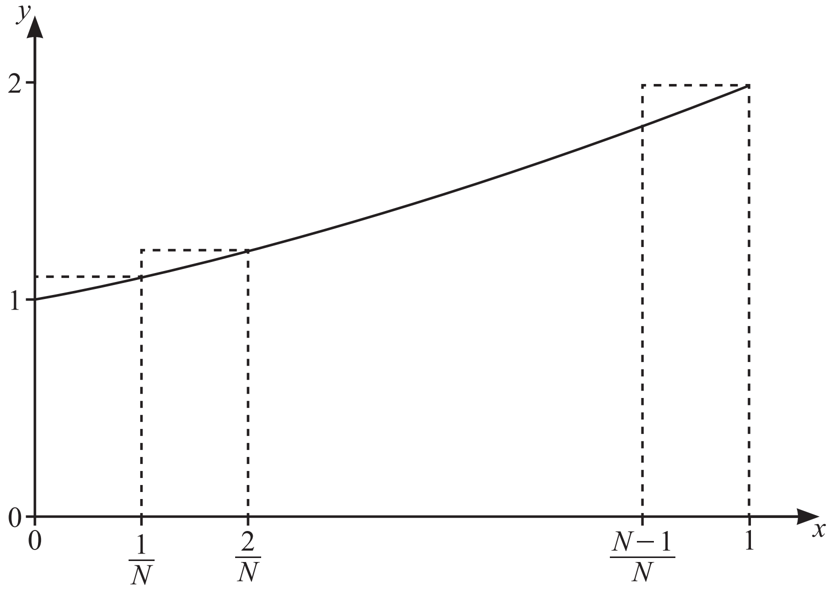
and verify that this equation has a root between 0.7 and 0.8. [4]

(b) By considering a suitable set of rectangles, use the diagram to show that

$$\sum_{r=2}^{n} r\,\text{sech}^2 r < n\tanh n + \ln\text{sech}\,n - \tanh 1 - \ln\text{sech}\,1.$$ [6]

15. [9231/s22/21/q4]

The diagram shows the curve with equation $y = 2^x$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.



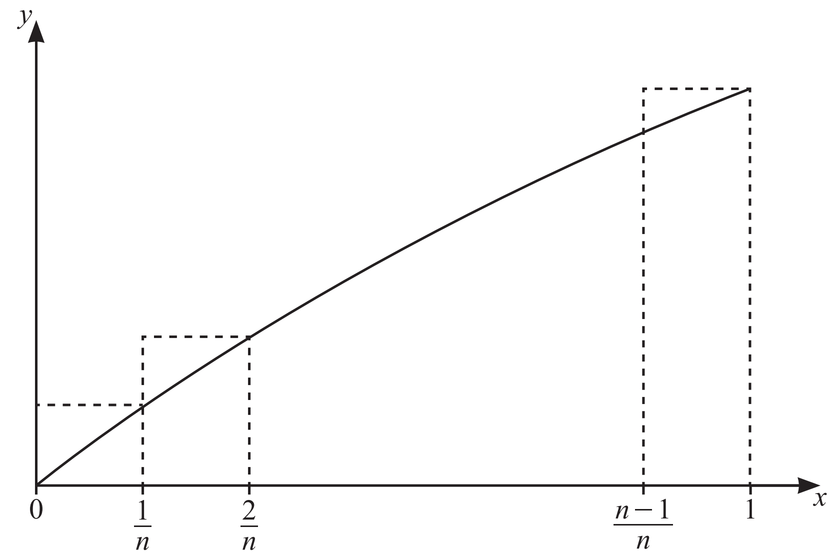
(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 2^x \, dx < U_N$, where

$$U_N = \frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}} - 1\right)}.$$ [4]

(b) Use a similar method to find, in terms of N, a lower bound L_N for $\int_0^1 2^x \, dx$. [4]

(c) Find the least value of N such that $U_N - L_N < 10^{-4}$. [2]

16. [9231/s22/23/q6]



The diagram shows the curve with equation $y = \ln(1+x)$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles each of width $\frac{1}{n}$.

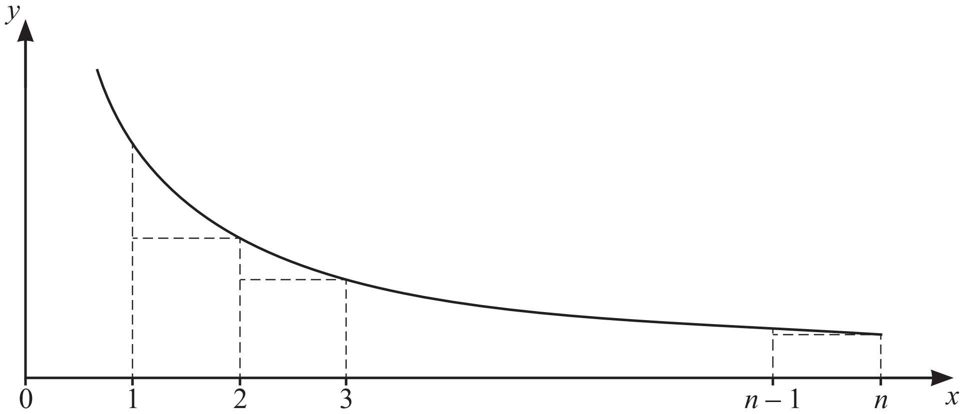
(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \ln(1+x)\,dx < U_n$, where

$$U_n = \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n.$$ [4]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \ln(1+x)\,dx$. [4]

(c) By simplifying $U_n - L_n$, show that $\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

17. [9231/w22/22/q6]

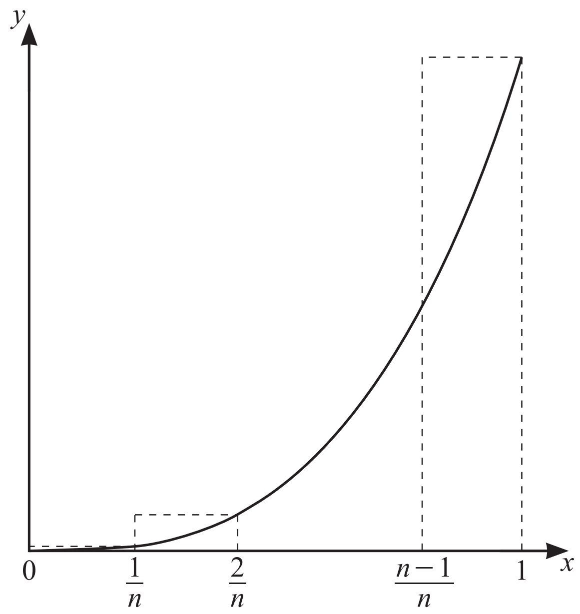


The diagram shows the curve $y = \dfrac{1}{\sqrt{x^2+2x}}$ for $x > 0$, together with a set of $(n-1)$ rectangles of unit width.

By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r^2+2r}} < \ln\left(n+1+\sqrt{n^2+2n}\right) + \tfrac{1}{3}\sqrt{3} - \ln\left(2+\sqrt{3}\right).$$ [10]

18. [9231/s21/21/q3]



The diagram shows the curve with equation $y = x^3$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

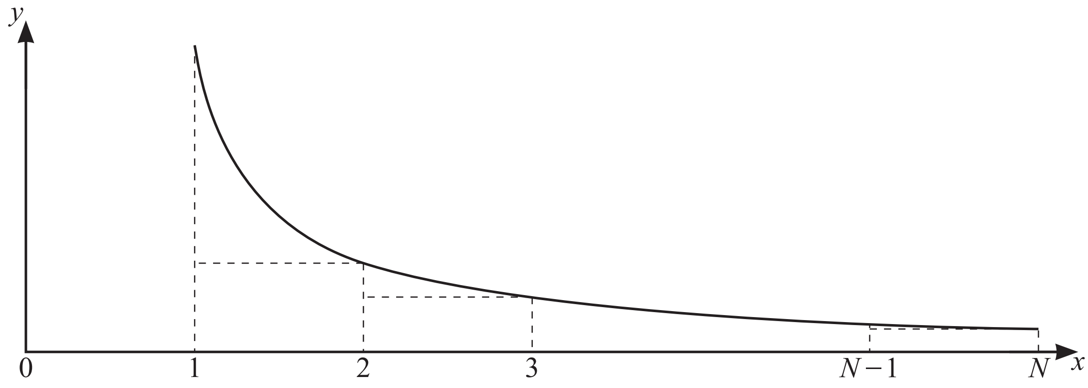
(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 x^3\,dx < U_n$, where

$$U_n = \left(\frac{n+1}{2n}\right)^2.$$ [4]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 x^3\,dx$. [4]

(c) Find the least value of n such that $U_n - L_n < 10^{-3}$. [2]

19. [9231/s21/23/q3]



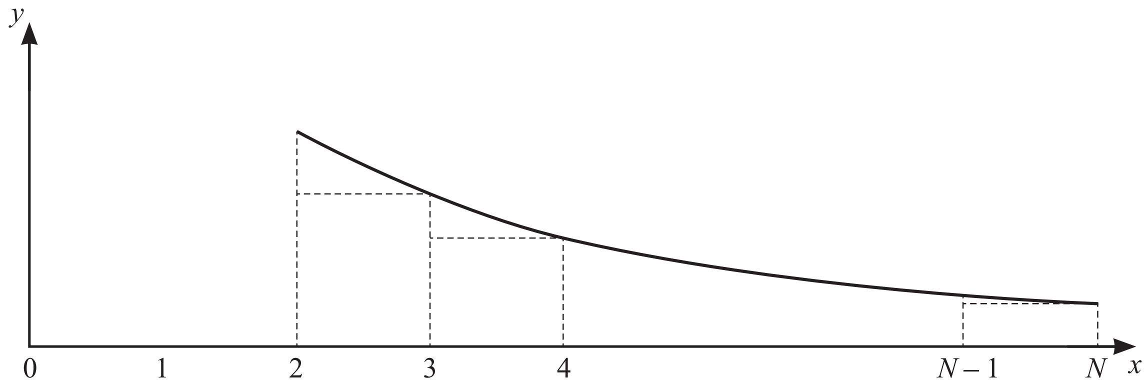
The diagram shows the curve $y = \frac{x}{2x^2-1}$ for $x \geqslant 1$, together with a set of $N-1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{r}{2r^2-1} < \frac{1}{4}\ln(2N^2-1)+1.$$ [7]

(b) Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N} \frac{r}{2r^2-1}$. [3]

20. [9231/w21/21/q4]



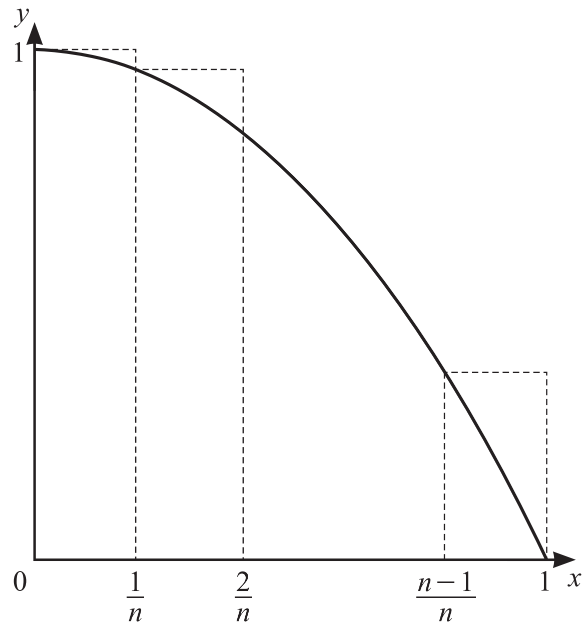
The diagram shows the curve with equation $y = \frac{\ln x}{x^2}$ for $x \geqslant 2$, together with a set of $(N-2)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{2+3\ln 2}{4} - \frac{1+\ln N}{N}.$$ [7]

(b) Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N} \frac{\ln r}{r^2}$. [3]

21. [9231/w21/22/q3]



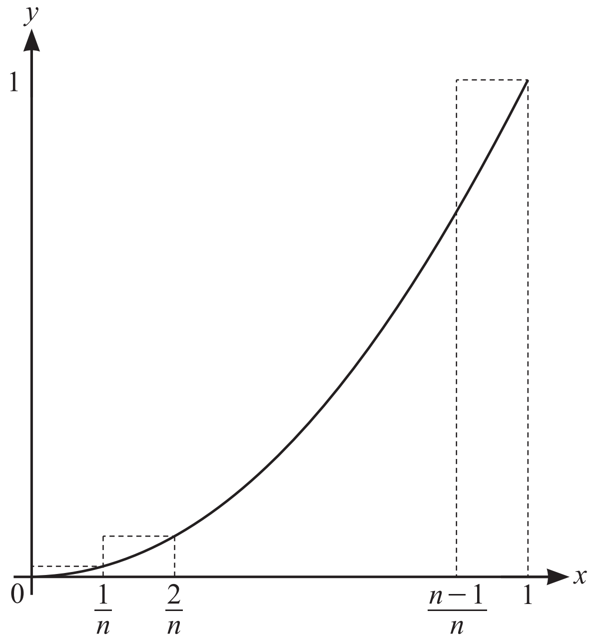
The diagram shows the curve with equation $y = 1 - x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1 - x^2)\,\mathrm{d}x < \frac{4n^2 + 3n - 1}{6n^2}. \qquad [4]$$

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1 - x^2)\,\mathrm{d}x$. [4]

22. [9231/s20/21/q4]



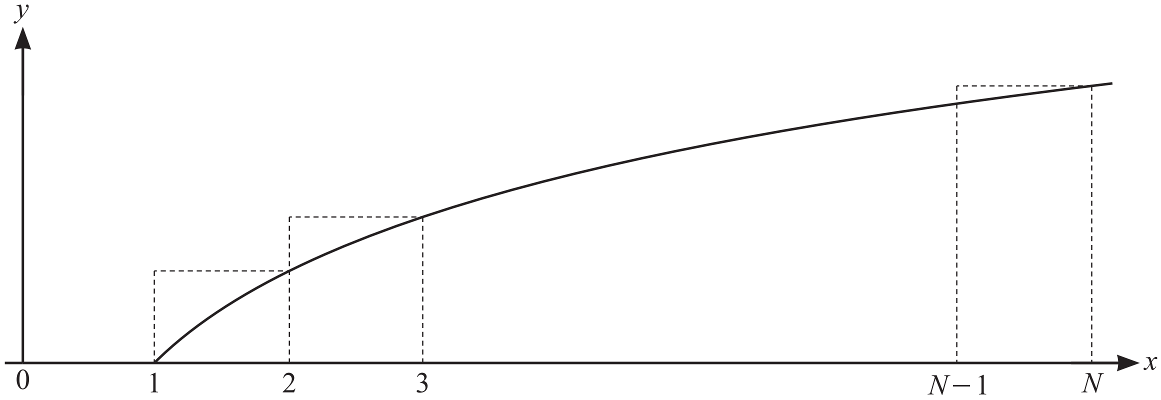
The diagram shows the curve with equation $y = x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 x^2\,dx < \frac{2n^2+3n+1}{6n^2}.$$ [4]

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 x^2\,dx$. [4]

23. [9231/s20/23/q4]



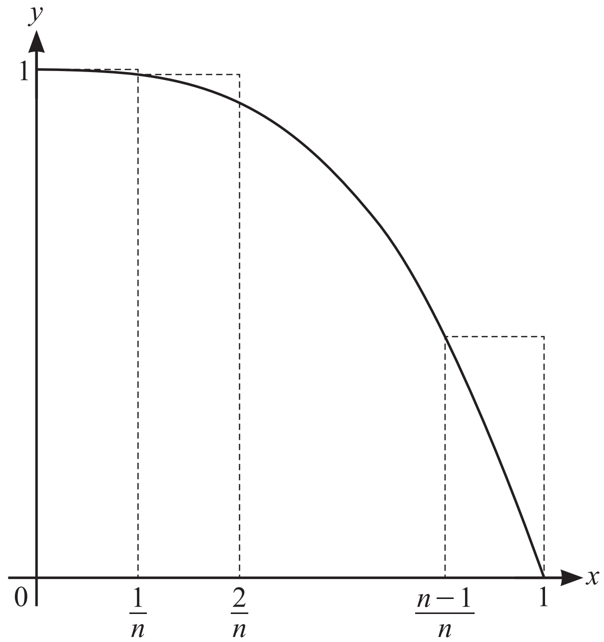
The diagram shows the curve with equation $y = \ln x$ for $x \geqslant 1$, together with a set of $(N-1)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\ln N! > N \ln N - N + 1. \qquad [5]$$

(b) Use a similar method to find, in terms of N, an upper bound for $\ln N!$. [3]

24. [9231/w20/21/q4]



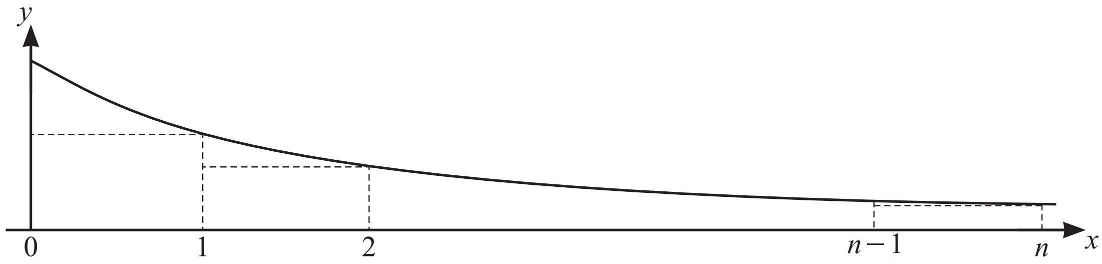
The diagram shows the curve with equation $y = 1 - x^3$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1-x^3)\,\mathrm{d}x \leqslant \frac{3n^2+2n-1}{4n^2}. \qquad [4]$$

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1-x^3)\,\mathrm{d}x$. [4]

25. [9231/w20/22/q8]



The diagram shows the curve $y = \dfrac{1}{\sqrt{x^2+x+1}}$ for $x \geqslant 0$, together with a set of n rectangles of unit width. By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r^2+r+1}} < \ln\left(\tfrac{1}{3} + \tfrac{2}{3}n + \tfrac{2}{3}\sqrt{n^2+n+1}\right).$$ [10]

Chapter 7

Complex numbers

1. [9231/s25/21/q1]

Find the roots of the equation $z^3 = 27 - 27\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

2. [9231/s25/21/q3]

By considering the binomial expansion of $\left(z-\frac{1}{z}\right)^5$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\operatorname{cosec}^5\theta=\frac{a}{\sin 5\theta+b\sin 3\theta+c\sin\theta},$$

where a, b and c are integers to be determined. [6]

3. [9231/s25/23/q5]

(a) Use de Moivre's theorem to show that

$$\sec 5\theta = \frac{\sec^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16}.$$ [6]

(b) Hence, obtain the roots of the equation

$$\sqrt{3}x^5 - 10x^4 + 40x^2 - 32 = 0$$

in the form $\sec(q\pi)$, where q is rational. [4]

4. [9231/s25/24/q5]

(a) Use de Moivre's theorem to show that

$$\sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta.$$ [5]

(b) Hence find all roots of the equation

$$64x^6 - 112x^4 + 56x^2 - 7 = 0$$

in the form $\sin q\pi$, where q is a rational number. [3]

5. [9231/w25/21/q8]

(a) State the sum of the series $z + z^2 + ... + z^n$, for $z \neq 1$. [1]

(b) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$ use de Moivre's theorem to deduce that

$$\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m \sin m\theta = \frac{\left(\frac{1}{2}\right)^{n+2}\sin n\theta - \left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta + \frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta}.$$ [6]

(c) Use the result in **(b)** to find $\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m m\cos m\theta$ in terms of n and θ. [You do not need to simplify your answer.] [3]

(d) Hence find $\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m m\cos m\theta$ in terms of $\cos\theta$. [You may assume that $\frac{n}{2^n} \to 0$ as $n \to \infty$.] [2]

6. [9231/w25/22/q2]

Find the roots of the equation $z^4 = 8 - 8\mathrm{i}\sqrt{3}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

7. [9231/w25/22/q4]

(a) By considering the binomial expansion of $\left(z+\frac{1}{z}\right)^6$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^6\theta = a\cos 6\theta + b\cos 4\theta + c\cos 2\theta + d,$$

where a, b, c and d are constants to be determined. [5]

(b) Find the exact value of $\int_0^{\frac{1}{2}\pi} \cos^6 2x \, \mathrm{d}x$. [3]

8. [9231/w25/24/q3]

(a) Use de Moivre's theorem to show that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta. \qquad [4]$$

(b) Hence obtain the roots of the equation

$$32x^5 - 40x^3 + 10x - \sqrt{3} = 0$$

in the form $\sin q\pi$, where q is a rational number. [4]

9. [9231/s24/21/q1]

Find the roots of the equation $z^3 = -108\sqrt{3} + 108\mathrm{i}$, giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $0 < \theta < 2\pi$. [5]

10. [9231/s24/23/q6]

(a) Show that $\sum_{r=1}^{n} z^{4r} = \frac{z^{4n+2} - z^2}{z^2 - z^{-2}}$, for $z^2 \neq z^{-2}$. [2]

(b) By letting $z = \cos\theta + \mathrm{i}\sin\theta$, show that, if $\sin 2\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(4r\theta) = \frac{\cos 2\theta - \cos(4n+2)\theta}{2\sin 2\theta}.$$ [5]

11. [9231/w24/21/q8]

(a) By considering the binomial expansion of $\left(z+\frac{1}{z}\right)^7$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^7\theta = a\cos 7\theta + b\cos 5\theta + c\cos 3\theta + d\cos\theta,$$

where a, b, c and d are constants to be determined. [5]

Let $I_n = \int_0^{\frac{1}{4}\pi} \cos^n\theta \,\mathrm{d}\theta$.

(b) Show that

$$nI_n = 2^{-\frac{1}{2}n} + (n-1)I_{n-2}. \quad [4]$$

(c) Using the results given in parts **(a)** and **(b)**, find the exact value of I_9. [5]

12. [9231/w24/22/q4]

(a) Use de Moivre's theorem to show that

$$\cot 6\theta = \frac{\cot^6\theta - 15\cot^4\theta + 15\cot^2\theta - 1}{6\cot^5\theta - 20\cot^3\theta + 6\cot\theta}.$$ [6]

(b) Hence obtain the roots of the equation

$$x^6 - 6x^5 - 15x^4 + 20x^3 + 15x^2 - 6x - 1 = 0$$

in the form $\cot(q\pi)$, where q is a rational number. [4]

13. [9231/s23/21/q3]

(a) By considering the binomial expansion of $(z+z^{-1})^4$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [5]

(b) Use the substitution $x = \sin\theta$ to find the exact value of $\int_0^{\frac{1}{2}} (1-x^2)^{\frac{3}{2}}\,\mathrm{d}x$. [3]

14. [9231/s23/23/q3]

By considering the binomial expansions of $\left(z+\frac{1}{z}\right)^4$ and $\left(z-\frac{1}{z}\right)^4$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cot^4\theta=\frac{\cos 4\theta+a\cos 2\theta+b}{\cos 4\theta-a\cos 2\theta+b},$$

where a and b are integers to be determined. [7]

15. [9231/w23/21/q2]

Find the roots of the equation $(z+5\mathrm{i})^3 = 4+4\sqrt{3}\,\mathrm{i}$, giving your answers in the form $r\cos\theta + \mathrm{i}(r\sin\theta - 5)$, where $r > 0$ and $0 < \theta < 2\pi$. [5]

16. [9231/w23/22/q3]

(a) Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \qquad [4]$$

(b) Hence obtain the roots of the equation

$$32x^5 - 40x^3 + 10x - \sqrt{2} = 0$$

in the form $\cos(q\pi)$, where q is a rational number. [4]

17. [9231/s22/21/q7]

(a) Use de Moivre's theorem to show that

$$\operatorname{cosec} 7\theta = \frac{\operatorname{cosec}^7\theta}{7\operatorname{cosec}^6\theta - 56\operatorname{cosec}^4\theta + 112\operatorname{cosec}^2\theta - 64}.$$

[6]

(b) Hence obtain the roots of the equation

$$x^7 - 14x^6 + 112x^4 - 224x^2 + 128 = 0$$

in the form $\operatorname{cosec} q\pi$, where q is rational. [5]

18. [9231/s22/23/q1]

Find the roots of the equation $z^3 = 7\sqrt{3} - 7\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

19. [9231/s22/23/q8.c]

(a) Find $\int \sin\theta \cos^n\theta \, d\theta$, where $n \neq -1$. [2]

Let $I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m\theta \cos^n\theta \, d\theta$.

(b) Show that, for $m \geqslant 2$ and $n \geqslant 0$,

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$ [5]

(c) By considering the binomial expansion of $\left(z + \frac{1}{z}\right)^5$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta,$$

where a, b and c are constants to be determined. [5]

(d) Using the results given in parts **(b)** and **(c)**, find the exact value of $I_{2,5}$. [4]

20. [9231/w22/21/q7]

(a) State the sum of the series $1+w+w^2+w^3+\ldots+w^{n-1}$, for $w \neq 1$. [1]

(b) Show that $(1+\mathrm{i}\tan\theta)^k = \sec^k\theta(\cos k\theta+\mathrm{i}\sin k\theta)$, where θ is not an integer multiple of $\frac{1}{2}\pi$. [2]

(c) By considering $\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k$, show that

$$\sum_{k=0}^{n-1}\sec^k\theta\sin k\theta = \cot\theta(1-\sec^n\theta\cos n\theta),$$

provided θ is not an integer multiple of $\frac{1}{2}\pi$. [5]

(d) Hence find $\sum_{k=0}^{6m-1}2^k\sin\left(\frac{1}{3}k\pi\right)$ in terms of m. [2]

21. [9231/w22/22/q5]

(a) Write down the fourth roots of unity. [1]

(b) Use de Moivre's theorem to show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$ [4]

(c) Hence obtain the real roots of the equation

$$16(8x^4 - 8x^2 + 1)^4 - 9 = 0$$

in the form $\cos(q\pi)$, where q is a rational number. [5]

22. [9231/s21/21/q5]

(a) State the sum of the series $z+z^2+z^3+...+z^n$, for $z \neq 1$. [1]

(b) Given that z is an nth root of unity and $z \neq 1$, deduce that $1+z+z^2+...+z^{n-1}=0$. [2]

(c) Given instead that $z=\frac{1}{3}(\cos\theta+\mathrm{i}\sin\theta)$, use de Moivre's theorem to show that

$$\sum_{m=1}^{\infty} 3^{-m}\cos m\theta = \frac{3\cos\theta-1}{10-6\cos\theta}.$$ [7]

23. [9231/s21/23/q1]

(a) Find a and b such that

$$z^8 - \mathrm{i}z^5 - z^3 + \mathrm{i} = (z^5 - a)(z^3 - b).$$ [1]

(b) Hence find the roots of

$$z^8 - \mathrm{i}z^5 - z^3 + \mathrm{i} = 0,$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [6]

24. [9231/s21/23/q4]

By considering the binomial expansions of $\left(z+\frac{1}{z}\right)^5$ and $\left(z-\frac{1}{z}\right)^5$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\tan^5\theta = \frac{\sin 5\theta - a\sin 3\theta + b\sin\theta}{\cos 5\theta + a\cos 3\theta + b\cos\theta},$$

where a and b are integers to be determined. [7]

25. [9231/w21/21/q6]

(a) Use de Moivre's theorem to show that

$$\operatorname{cosec} 5\theta = \frac{\operatorname{cosec}^5\theta}{5\operatorname{cosec}^4\theta - 20\operatorname{cosec}^2\theta + 16}. \qquad [6]$$

(b) Hence obtain the roots of the equation

$$x^5 - 10x^4 + 40x^2 - 32 = 0$$

in the form $\operatorname{cosec}(q\pi)$, where q is rational. [4]

26. [9231/w21/22/q4]

(a) Write down all the roots of the equation $x^5 - 1 = 0$. [2]

(b) Use de Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. [4]

(c) Use the results of parts **(a)** and **(b)** to express each real root of the equation

$$8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = 0$$

in the form $\cos k\pi$, where k is a rational number. [4]

27. [9231/s20/21/q3]

(a) Find the roots of the equation $z^3 = -1 - \mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [5]

Let $w = z_1^{3k} + z_2^{3k} + z_3^{3k}$, where k is a positive integer and z_1, z_2, z_3 are the roots of $z^3 = -1 - \mathrm{i}$.

(b) Express w in the form $R\mathrm{e}^{\mathrm{i}\alpha}$, where $R > 0$, giving R and α in terms of k. [3]

28. [9231/s20/23/q8]

(a) Use de Moivre's theorem to show that $\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$. [6]

It is given that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

(b) Find the exact value of $\int_0^{\frac{1}{3}\pi}\left(\cos^6(\tfrac{1}{4}x) + \sin^6(\tfrac{1}{4}x)\right)dx$. [4]

(c) Express each root of the equation $16c^6 + 16\left(1 - c^2\right)^3 - 13 = 0$ in the form $\cos k\pi$, where k is a rational number. [5]

29. [9231/w20/21/q6.a]

(a) Use de Moivre's theorem to show that $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$. [5]

(b) Find the solution of the differential equation

$$\frac{dy}{d\theta} + y\cot\theta = \sin^3\theta$$

for which $y = 0$ when $\theta = \frac{1}{2}\pi$. [6]

30. [9231/w20/22/q3]

Find all the roots of the equation $(w+1)^6 = 1$, giving your answers in the form $x + \mathrm{i}y$ where x and y are real and exact. [4]

31. [9231/w20/22/q7]

(a) Show that $\sum_{r=1}^{n} z^{2r} = \frac{z^{2n+1} - z}{z - z^{-1}}$, for $z \neq 0, 1, -1$. [2]

(b) By letting $z = \cos\theta + \mathrm{i}\sin\theta$, show that, if $\sin\theta \neq 0$,

$$1 + 2\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin(2n+1)\theta}{\sin\theta}.$$ [5]

32. [9231/s19/11/q8]

(i) Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$ [5]

33. [9231/s19/13/q3]

(i) Write down the fifth roots of unity. [2]

(ii) Find all the roots of the equation

$$z^{10} + z^5 + 1 = 0,$$

giving each root in the form $e^{i\theta}$. [5]

34. [9231/w19/11/q9]

(i) Use de Moivre's theorem to show that

$$\sec 6\theta = \frac{\sec^6 \theta}{32 - 48\sec^2 \theta + 18\sec^4 \theta - \sec^6 \theta}. \qquad [6]$$

(ii) Hence obtain the roots of the equation

$$3x^6 - 36x^4 + 96x^2 - 64 = 0$$

in the form $\sec q\pi$, where q is rational. [5]

35. [9231/s18/11/q11e]

(i) Show that if $z = e^{i\theta}$ and $z \neq -1$ then

$$\frac{z-1}{z+1} = i\tan\tfrac{1}{2}\theta. \qquad [3]$$

(ii) Hence, or otherwise, show that if z is a cube root of unity then

$$\frac{z^3-1}{z^3+1} + \frac{z^2-1}{z^2+1} + \frac{z-1}{z+1} = 0. \qquad [5]$$

(iii) Hence write down three roots of the equation

$$(z^3-1)(z^2+1)(z+1) + (z^2-1)(z^3+1)(z+1) + (z-1)(z^3+1)(z^2+1) = 0$$

and find the other three roots. Give your answers in an exact form. [6]

36. [9231/s18/13/q3]

(i) Use de Moivre's theorem to show that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$ [3]

(ii) Hence find all the roots of the equation

$$x^4 - 6x^2 + 1 = 0$$

in the form $\tan q\pi$, where q is a positive rational number. [5]

37. [9231/w18/11/q7]

(i) Use de Moivre's theorem to show that

$$\sin 8\theta = 8\sin\theta\cos\theta(1 - 10\sin^2\theta + 24\sin^4\theta - 16\sin^6\theta).$$ [6]

(ii) Use the equation $\dfrac{\sin 8\theta}{\sin 2\theta} = 0$ to find the roots of

$$16x^6 - 24x^4 + 10x^2 - 1 = 0$$

in the form $\sin k\pi$, where k is rational. [4]

38. [9231/w18/12/q8]

(i) By considering the binomial expansion of $\left(z + \dfrac{1}{z}\right)^6$, where $z = \cos\theta + \mathrm{i}\sin\theta$, express $\cos^6\theta$ in the form

$$\tfrac{1}{32}(p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta),$$

where p, q, r and s are integers to be determined. [6]

(ii) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^6\left(\tfrac{1}{2}x\right)\,\mathrm{d}x.$$ [4]

39. [9231/s17/11/q8]

(i) Let $z = \cos\theta + \mathrm{i}\sin\theta$. Show that $z - \dfrac{1}{z} = 2\mathrm{i}\sin\theta$ and hence express $16\sin^5\theta$ in the form $\sin 5\theta + p\sin 3\theta + q\sin\theta$, where p and q are integers to be determined. [6]

(ii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{3}\pi} 16\sin^5\theta\,\mathrm{d}\theta$. [3]

40. [9231/s17/13/q7]

(i) Use de Moivre's theorem to prove that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$ [5]

(ii) Hence find the solutions of the equation

$$t^4 - 4t^3 - 6t^2 + 4t + 1 = 0,$$

giving your answers in the form $\tan k\pi$, where k is a rational number. [5]

41. [9231/w17/11/q10]

(i) Use de Moivre's theorem to show that

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta. \qquad [5]$$

(ii) Hence explain why the roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \pm\sin\frac{1}{5}\pi$ and $x = \pm\sin\frac{2}{5}\pi$. [3]

(iii) Without using a calculator, find the exact values of

$$\sin\tfrac{1}{5}\pi\sin\tfrac{2}{5}\pi\sin\tfrac{3}{5}\pi\sin\tfrac{4}{5}\pi \quad \text{and} \quad \sin^2\left(\tfrac{1}{5}\pi\right) + \sin^2\left(\tfrac{2}{5}\pi\right). \qquad [4]$$

42. [9231/s16/11/q6]

Use de Moivre's theorem to express $\cot 7\theta$ in terms of $\cot \theta$. [4]

Use the equation $\cot 7\theta = 0$ to show that the roots of the equation

$$x^6 - 21x^4 + 35x^2 - 7 = 0$$

are $\cot\left(\frac{1}{14}k\pi\right)$ for $k = 1, 3, 5, 9, 11, 13$, and deduce that

$$\cot^2\left(\tfrac{1}{14}\pi\right)\cot^2\left(\tfrac{3}{14}\pi\right)\cot^2\left(\tfrac{5}{14}\pi\right) = 7.$$ [5]

43. [9231/s16/13/q9]

Use de Moivre's theorem to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [4]

Find the corresponding expression for $\sin^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$. [4]

Hence find the exact value of $\int_0^{\frac{1}{8}\pi} (\cos^4\theta + \sin^4\theta)\,d\theta$. [3]

44. [9231/w16/11/q10]

Let $z = \cos\theta + \mathrm{i}\sin\theta$. Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta. \qquad [2]$$

By considering $\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$, show that

$$\sin^4\theta\cos^2\theta = \tfrac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2). \qquad [7]$$

Hence find the exact value of $\int_0^{\frac{1}{4}\pi} \sin^4\theta\cos^2\theta\,\mathrm{d}\theta$. [3]

45. [9231/s15/11/q8]

By considering $\sum_{r=1}^{n} z^{2r-1}$, where $z = \cos\theta + \mathrm{i}\sin\theta$, show that, if $\sin\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}.$$ [7]

Deduce that

$$\sum_{r=1}^{n}(2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right] = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right).$$ [4]

46. [9231/s15/13/q6]

Let $z = \cos\theta + \mathrm{i}\sin\theta$. Use the binomial expansion of $(1+z)^{\boldsymbol{n}}$, where $\boldsymbol{n}$ is a positive integer, to show that

$$\binom{\boldsymbol{n}}{1}\cos\theta + \binom{\boldsymbol{n}}{2}\cos 2\theta + \ldots + \binom{\boldsymbol{n}}{\boldsymbol{n}}\cos \boldsymbol{n}\theta = 2^{\boldsymbol{n}}\cos^{\boldsymbol{n}}\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\boldsymbol{n}\theta\right) - 1. \qquad [7]$$

Find

$$\binom{\boldsymbol{n}}{1}\sin\theta + \binom{\boldsymbol{n}}{2}\sin 2\theta + \ldots + \binom{\boldsymbol{n}}{\boldsymbol{n}}\sin \boldsymbol{n}\theta. \qquad [2]$$

47. [9231/w15/11/q10]

Using de Moivre's theorem, show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$ [5]

Hence show that the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$. [4]

Deduce a quadratic equation, with integer coefficients, having roots $\sec^2\left(\frac{1}{5}\pi\right)$ and $\sec^2\left(\frac{2}{5}\pi\right)$. [3]

Chapter 8

Differential equations

1. [9231/s25/21/q5]

Find the particular solution of the differential equation

$$6\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 6x = e^{-t},$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [10]

2. [9231/s25/21/q7]

Find the solution of the differential equation

$$\frac{dy}{dx} - \frac{x+5}{x^2+10x+61}y = 1,$$

given that $y = 0$ when $x = 3$. Give your answer in an exact form. [10]

3. [9231/s25/23/q4]

Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 2t^2 + t - 1,$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [10]

4. [9231/s25/23/q7]

Find the solution of the differential equation

$$\frac{dy}{dx} - \frac{2x+6}{x^2+6x+5}y = 4,$$

given that $y = 0$ when $x = 0$. Give your answer in an exact form. [9]

5. [9231/s25/24/q3]

Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 13e^{3x}$$

given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. [10]

6. [9231/s25/24/q6]

Find the solution of the differential equation

$$x\frac{dy}{dx} - y = 2x^2 \tan^{-1} x$$

for which $y = \frac{1}{2}\pi$ when $x = 1$. Give your answer in the form $y = f(x)$. [9]

7. [9231/w25/21/q4]

Find the particular solution of the differential equation

$$5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 5x + 3,$$

given that, when $x = 0$, $y = \frac{dy}{dx} = 0$. [10]

8. [9231/w25/21/q7]

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1}x\right) = \frac{1}{1-x^2}$. [3]

(b) Find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2\tanh^{-1}x,$$

for $0 < x < 1$, given that $y = 0$ when $x = \frac{1}{2}$. Give your answer in an exact form. [9]

9. [9231/w25/22/q5]

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 5\cos x. \qquad [7]$$

(b) For large positive values of x and for any initial conditions, show that the solution to part **(a)** can be approximated by

$$y \approx R\sin(x + \phi),$$

where the constants R and ϕ are to be determined. [3]

10. [9231/w25/22/q7]

(a) Show that $\frac{d}{dx}\left(\frac{1}{2}x\sqrt{4-x^2}+2\sin^{-1}\left(\frac{1}{2}x\right)\right)=\sqrt{4-x^2}$. [3]

(b) Find the solution of the differential equation

$$2\frac{dy}{dx}+\frac{y}{2+x}=2\sqrt{2-x}$$

for which $y=\frac{1}{2}$ when $x=1$. Give your answer in an exact form. [8]

11. [9231/w25/24/q5]

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y = \sin 2x+\cos 2x.$$ [7]

(b) For large positive values of x and for any initial conditions, show that the solution to part **(a)** can be approximated by

$$y \backsimeq R\sin(2x-\phi),$$

where R and ϕ are positive constants to be determined. [3]

12. [9231/w25/24/q6]

Find the particular solution of the differential equation

$$\frac{dy}{dx} + \frac{1+x}{1-2x-x^2}y = 1$$

given that $y = \pi$ when $x = 0$. Give your answer in an exact form. [10]

13. [9231/s24/21/q6]

(a) Show that $\left(\cosh x+\sinh x\right)^{\frac{1}{2}} = \mathrm{e}^{\frac{1}{2}x}$. [2]

(b) Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+3y = 5\left(\cosh x+\sinh x\right)^{\frac{1}{2}},$$

given that, when $x = 0,\ y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{3}$. [10]

14. [9231/s24/21/q7]

(a) Use the substitution $u = 1 + x^2$ to find

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x. \qquad [2]$$

(b) Find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2 \sinh^{-1} x,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = \mathrm{f}(x)$. [10]

15. [9231/s24/23/q5]

(a) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 25x = 338\sin t.$$ [7]

(b) Show that, for large positive values of t and for any initial conditions,

$$x \approx R\sin(t - \phi),$$

where the constants R and ϕ are to be determined. [3]

16. [9231/s24/23/q7]

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\cosh^{-1}\frac{x}{3}\right)=\sqrt{x^2-9}.$$ [3]

(b) Find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}-y=x^2\sqrt{x^2-9},$$

given that $y=1$ when $x=3$. Give your answer in the form $y=\mathrm{f}(x)$. [9]

17. [9231/w24/21/q5]

Find the particular solution of the differential equation

$$6\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + x = t^2 + t + 1,$$

given that, when $t = 0$, $x = 12$ and $\frac{dx}{dt} = -6$. [10]

18. [9231/w24/21/q7]

(a) Show that an appropriate integrating factor for

$$\sqrt{x^2+16}\,\frac{dy}{dx}+y=x\sqrt{x^2+16}$$

is $\frac{1}{4}x+\frac{1}{4}\sqrt{x^2+16}$. [4]

(b) Hence find the solution of the differential equation

$$\sqrt{x^2+16}\,\frac{dy}{dx}+y=x\sqrt{x^2+16}$$

for which $y=6$ when $x=3$. [6]

19. [9231/w24/22/q5]

Find the particular solution of the differential equation

$$3\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x^2,$$

given that, when $x=0$, $y=\frac{dy}{dx}=0$. [10]

20. [9231/w24/22/q7]

(a) Show that $\frac{d}{dx}(\ln(\tanh x)) = 2\operatorname{cosech} 2x$. [3]

(b) Find the solution of the differential equation

$$\sinh 2x \frac{dy}{dx} + 2y = \sinh 2x$$

for which $y = 5$ when $x = \ln 2$. Give your answer in an exact form. [7]

21. [9231/s23/21/q2]

Use the substitution $z = x + y$ to find the solution of the differential equation

$$\frac{dy}{dx} = \frac{1 + 3x + 3y}{3x + 3y - 1}$$

for which $y = 0$ when $x = 1$. Give your answer in the form $a\ln(x+y) + b(x-y) + c = 0$, where a, b and c are constants to be determined. [7]

22. [9231/s23/21/q6]

Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 36x = 37\sin t,$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [11]

23. [9231/s23/23/q2]

The variables x and y are related by the differential equation

$$6\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + x = t^2 + 10t + 13.$$

(a) Find the general solution for x in terms of t. [6]

(b) State an approximate solution for large positive values of t. [1]

24. [9231/s23/23/q5]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\cosh^2 x = \cosh 2x + 1.$$ [3]

(b) Find the solution of the differential equation

$$\frac{dy}{dx} + 2y\tanh x = 1$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. [8]

25. [9231/w23/21/q4]

Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 27x^2,$$

given that, when $x = 0$, $y = 2$ and $\frac{dy}{dx} = -8$. [10]

26. [9231/w23/21/q6]

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\sinh 2x = 2\sinh x \cosh x.$$ [3]

(b) Using the substitution $u = \sinh x$, find $\int \sinh^2 2x \cosh x \, dx$. [4]

(c) Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\tanh x = \sinh^2 2x,$$

given that $y = 4$ when $x = 0$. Give your answer in the form $y = f(x)$. [7]

27. [9231/w23/22/q4]

Find the solution of the differential equation

$$\frac{dy}{dx} + 3y = \sin x$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. [9]

28. [9231/w23/22/q8]

It is given that $v = y^4$ and

$$y^3 \frac{d^2y}{dx^2} + 3y^2\left(\frac{dy}{dx}\right)^2 + y^3\frac{dy}{dx} + y^4 = e^{-2x}.$$

(a) Show that

$$\frac{d^2v}{dx^2} + \frac{dv}{dx} + 4v = 4e^{-2x}.$$ [4]

(b) Find y in terms of x, given that, when $x = 0$, $y = 1$ and $\frac{dy}{dx} = -\frac{3}{8}$. [10]

29. [9231/s22/21/q3]

The variables t and x are related by the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = t^2 + 1.$$

(a) Find the general solution for x in terms of t. [6]

(b) Deduce an approximate value of $\frac{d^2x}{dt^2}$ for large positive values of t. [2]

30. [9231/s22/21/q6]

Use the substitution $y = vx$ to find the solution of the differential equation

$$x\frac{dy}{dx} = y + \sqrt{9x^2 + y^2}$$

for which $y = 0$ when $x = 1$. Give your answer in the form $y = f(x)$, where $f(x)$ is a polynomial in x.
[10]

31. [9231/s22/23/q5]

Find the solution of the differential equation

$$x(x+7)\frac{dy}{dx}+7y=x$$

for which $y=7$ when $x=1$. Give your answer in the form $y=f(x)$. [9]

32. [9231/s22/23/q7]

The variables x and y are related by the differential equation

$$4\frac{d^2y}{dx^2} - y = 3.$$

It is given that, when $x = 0$, $y = -3$ and $\frac{dy}{dx} = 2$.

(a) Find y in terms of x. [8]

(b) Deduce the exact value of x for which $y = 0$. Give your answer in logarithmic form. [3]

33. [9231/w22/21/q5]

Find the particular solution of the differential equation

$$2\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = 4x^2+3x+3,$$

given that, when $x = 0$, $y = \frac{dy}{dx} = 0$. [10]

34. [9231/w22/21/q8]

(a) Use the substitution $u = 1-(\theta-1)^2$ to find

$$\int \frac{\theta-1}{\sqrt{1-(\theta-1)^2}}\,d\theta. \qquad [3]$$

(b) Find the solution of the differential equation

$$\theta\frac{dy}{d\theta} - y = \theta^2 \sin^{-1}(\theta-1),$$

where $0 < \theta < 2$, given that $y = 1$ when $\theta = 1$. Give your answer in the form $y = f(\theta)$. [11]

35. [9231/w22/22/q4]

Find the solution of the differential equation

$$(4t^2-1)\frac{dx}{dt}+4x=4t^2-1$$

for which $x=3$ when $t=1$. Give your answer in the form $x=f(t)$. [9]

36. [9231/w22/22/q8]

It is given that $y = \cosh u$, where $u > 0$, and

$$\sqrt{\cosh^2 u - 1}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x}\right) + \cosh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 2\cosh u = 4\mathrm{e}^{-x}.$$

(a) Show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 4\mathrm{e}^{-x}.$$ [4]

(b) Find u in terms of x, given that, when $x = 0$, $u = \ln 3$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$. [10]

37. [9231/s21/21/q2]

The variables x and y are related by the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x + 1.$$

(a) Find the general solution for y in terms of x. [6]

(b) State an approximate solution for large positive values of x. [1]

38. [9231/s21/21/q4]

Find the solution of the differential equation

$$\sin\theta\frac{\mathrm{d}y}{\mathrm{d}\theta}+y=\tan\tfrac{1}{2}\theta,$$

where $0<\theta<\pi$, given that $y=1$ when $\theta=\frac{1}{2}\pi$. Give your answer in the form $y=\mathrm{f}(\theta)$. [9]

[You may use without proof the result that $\int \operatorname{cosec}\theta\,\mathrm{d}\theta=\ln\tan\frac{1}{2}\theta$.]

39. [9231/s21/23/q5]

The variables x and y are related by the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 4e^{-x}.$$

(a) Find the value of the constant k such that $y = kxe^{-x}$ is a particular integral of the differential equation. [4]

(b) Find the solution of the differential equation for which $y = \frac{dy}{dx} = \frac{1}{2}$ when $x = 0$. [6]

40. [9231/s21/23/q6.b]

(a) Starting from the definitions of sinh and cosh in terms of exponentials, prove that

$$2\sinh^2 x = \cosh 2x - 1. \qquad [3]$$

(b) Find the solution to the differential equation

$$\frac{dy}{dx} + y\coth x = 4\sinh x$$

for which $y = 1$ when $x = \ln 3$. [7]

41. [9231/w21/21/q5]

Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4\cos x,$$

given that, when $x = 0$, $y = -4$ and $\frac{dy}{dx} = 3$. [11]

42. [9231/w21/21/q7]

(a) Show that an appropriate integrating factor for

$$\sqrt{x^2-1}\frac{dy}{dx}+y=x^2-x\sqrt{x^2-1}$$

is $x+\sqrt{x^2-1}$. [4]

(b) Hence find the solution of the differential equation

$$\sqrt{x^2-1}\frac{dy}{dx}+y=x^2-x\sqrt{x^2-1}$$

for which $y=1$ when $x=\frac{5}{4}$. Give your answer in the form $y=f(x)$. [7]

43. [9231/w21/22/q2]

Find the solution of the differential equation

$$\frac{dy}{dx} + \frac{4x^3 y}{x^4 + 5} = 6x$$

for which $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$. [7]

44. [9231/w21/22/q7]

It is given that $y = x^2 w$ and

$$x^2 \frac{d^2w}{dx^2} + 4x(x+1)\frac{dw}{dx} + (5x^2 + 8x + 2)w = 5x^2 + 4x + 2.$$

(a) Show that

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5x^2 + 4x + 2.$$ [4]

(b) Find the general solution for w in terms of x. [7]

45. [9231/s20/21/q1]

Find the solution of the differential equation

$$\frac{dy}{dx} + 5y = e^{-7x}$$

for which $y = 0$ when $x = 0$. Give your answer in the form $y = f(x)$. [6]

46. [9231/s20/21/q7]

It is given that $x = t^3 y$ and

$$t^3 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \left(4t^3 + 6t^2\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \left(13t^3 + 12t^2 + 6t\right)y = 61\mathrm{e}^{\frac{1}{2}t}.$$

(a) Show that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 61\mathrm{e}^{\frac{1}{2}t}.$$ [4]

(b) Find the general solution for y in terms of t. [7]

47. [9231/s20/23/q1]

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} - 8\frac{dx}{dt} - 9x = 9e^{8t}.$$ [6]

48. **[9231/s20/23/q7]**

(a) Show that an appropriate integrating factor for

$$(x^2+1)\frac{dy}{dx}+y\sqrt{x^2+1}=x^2-x\sqrt{x^2+1}$$

is $x+\sqrt{x^2+1}$. [4]

(b) Hence find the solution of the differential equation

$$(x^2+1)\frac{dy}{dx}+y\sqrt{x^2+1}=x^2-x\sqrt{x^2+1}$$

for which $y=\ln 2$ when $x=0$. Give your answer in the form $y=f(x)$. [7]

49. [9231/w20/21/q2]

The variables x and y are related by the differential equation

$$9\frac{d^2y}{dx^2}+6\frac{dy}{dx}+y=3x^2+30x.$$

(a) Find the general solution for y in terms of x. [6]

(b) State an approximate solution for large positive values of x. [1]

50. [9231/w20/21/q6.b]

(a) Use de Moivre's theorem to show that $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$. [5]

(b) Find the solution of the differential equation

$$\frac{dy}{d\theta} + y\cot\theta = \sin^3\theta$$

for which $y = 0$ when $\theta = \frac{1}{2}\pi$. [6]

51. [9231/w20/22/q4]

Find the solution of the differential equation

$$x\frac{dy}{dx} + 2y = e^x$$

for which $y = 3$ when $x = 1$. Give your answer in the form $y = f(x)$. [8]

52. [9231/w20/22/q6]

Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 15x = 102\cos 3t,$$

given that, when $t = 0$, $x = 1$ and $\frac{dx}{dt} = 0$. [11]

53. [9231/s19/11/q7]

Find the particular solution of the differential equation

$$10\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = t + 2,$$

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$. [10]

54. [9231/s19/13/q8]

Find the particular solution of the differential equation

$$9\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + x = 50\sin t,$$

given that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. [10]

55. [9231/w19/11/q11e]

Answer only **one** of the following two alternatives.

EITHER

It is given that $w = \cos y$ and

$$\tan y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2\tan y \frac{dy}{dx} = 1 + e^{-2x} \sec y.$$

(i) Show that

$$\frac{d^2w}{dx^2} + 2\frac{dw}{dx} + w = -e^{-2x}.$$ [4]

(ii) Find the particular solution for y in terms of x, given that when $x = 0$, $y = \frac{1}{3}\pi$ and $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$. [10]

56. [9231/s18/11/q7]

Find the particular solution of the differential equation

$$49\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + y = 49x + 735,$$

given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$. [10]

57. [9231/s18/13/q10]

It is given that $t \neq 0$ and

$$t\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 9tx = 3t^2 + 1.$$

(i) Show that if $y = tx$ then

$$\frac{d^2y}{dt^2} + 9y = 3t^2 + 1. \qquad [3]$$

(ii) Find x in terms of t, given that $x = \frac{1}{9}\pi$ and $\frac{dx}{dt} = \frac{2}{3}$ when $t = \frac{1}{3}\pi$. [9]

58. [9231/w18/11/q10]

(i) Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 37 \sin 3t,$$

given that $x = 3$ and $\frac{dx}{dt} = 0$ when $t = 0$. [10]

(ii) Show that, for large positive values of t and for any initial conditions,

$$x \approx \sqrt{(37)} \sin(3t - \phi),$$

where the constant ϕ is such that $\tan \phi = 6$. [3]

59. [9231/w18/12/q4]

(i) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 4\sin t.$$ [7]

(ii) State an approximate solution for large positive values of t. [1]

60. [9231/s17/11/q10]

It is given that $x = t^{\frac{1}{2}}$, where $x > 0$ and $t > 0$, and y is a function of x.

(i) Show that $\dfrac{dy}{dx} = 2t^{\frac{1}{2}}\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dt} + 4t\dfrac{d^2y}{dt^2}$. [3]

(ii) Hence show that the differential equation

$$\frac{d^2y}{dx^2} - \left(8x + \frac{1}{x}\right)\frac{dy}{dx} + 12x^2y = 4x^2e^{-x^2} \qquad (*)$$

reduces to the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t}. \qquad [1]$$

(iii) Find the general solution of $(*)$, giving y in terms of x. [7]

61. [9231/s17/13/q8]

Find the solution of the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 18t^2 + 6t + 1,$$

given that, when $t = 0$, $x = 3$ and $\frac{dx}{dt} = 0$. [10]

62. [9231/w17/11/q2]

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 4 - 5t^2. \qquad [6]$$

63. [9231/s16/11/q9]

Find the value of the constant k such that $y = kx^2e^{2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}. \qquad (*)$$ [4]

Hence find the general solution of $(*)$. [3]

Find the particular solution of $(*)$ such that $y = 3$ and $\frac{dy}{dx} = -2$ when $x = 0$. [4]

64. [9231/s16/13/q10]

Given that y is a function of x and that $x = e^u$, show that

$$x\frac{dy}{dx} = \frac{dy}{du} \qquad \text{and} \qquad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}.$$ [3]

Given also that

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 17y = 34\ln x + 21,$$

deduce that

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + 17y = 34u + 21.$$ [1]

Find y in terms of x given that $y = 0$ and $\frac{dy}{dx} = -1$ when $x = 1$. [9]

65. [9231/w16/11/q6]

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 10x = 116 \sin 2t.$$ [8]

State an approximate solution for large positive values of t. [1]

66. [9231/s15/11/q11e]

Show that the substitution $v = \dfrac{1}{y}$ reduces the differential equation

$$\frac{2}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y^2}\frac{d^2y}{dx^2} - \frac{2}{y^2}\frac{dy}{dx} + \frac{5}{y} = 17 + 6x - 5x^2$$

to the differential equation

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + 5v = 17 + 6x - 5x^2.$$ [4]

Hence find y in terms of x, given that when $x = 0$, $y = \frac{1}{2}$ and $\dfrac{dy}{dx} = -1$. [10]

67. [9231/s15/13/q9]

Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 10x = 2\sin t - 3\cos t,$$

given that, when $t = 0$, $x = 3.3$ and $\frac{dx}{dt} = 0.9$. [11]

68. [9231/w15/11/q2]

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 7 - 2t^2. \qquad [6]$$

Formula Sheet MF19

Cambridge Assessment
International Education

List MF19

List of formulae and statistical tables

Cambridge International AS & A Level
Mathematics (9709) and Further Mathematics (9231)

For use from 2020 in all papers for the above syllabuses.

CST319

2508709701

Edited by Thoridal

PURE MATHEMATICS

Mensuration

Volume of sphere = $\frac{4}{3}\pi r^3$

Surface area of sphere = $4\pi r^2$

Volume of cone or pyramid = $\frac{1}{3}\times$ base area $\times$ height

Area of curved surface of cone = $\pi r \times$ slant height

Arc length of circle $= r\theta$ (θ in radians)

Area of sector of circle $= \frac{1}{2}r^2\theta$ (θ in radians)

Algebra

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d\ , \qquad S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}, \qquad S_n = \frac{a(1-r^n)}{1-r}\ (r \neq 1), \qquad S_\infty = \frac{a}{1-r}\ \left(|r| < 1\right)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$, where n is a positive integer

and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$, where n is rational and $|x| < 1$

Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1+\tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1} x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
uv	$v\frac{du}{dx} + u\frac{dv}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$	
x^n	$\dfrac{x^{n+1}}{n+1}$	$(n \neq -1)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$	$(\lvert x\rvert < a)$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$$

Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

FURTHER PURE MATHEMATICS

Algebra

Summations:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Trigonometry

If $t = \tan\frac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x \equiv 1, \qquad \sinh 2x \equiv 2\sinh x \cosh x, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$	
$\sec x$	$\ln\lvert \sec x + \tan x\rvert = \ln\lvert \tan(\frac{1}{2}x + \frac{1}{4}\pi)\rvert$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\operatorname{cosec} x$	$-\ln\lvert \operatorname{cosec} x + \cot x\rvert = \ln\lvert \tan(\frac{1}{2}x)\rvert$	$(0 < x < \pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	$(x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	

MECHANICS

Uniformly accelerated motion

$$v = u + at\,, \qquad s = \tfrac{1}{2}(u+v)t\,, \qquad s = ut + \tfrac{1}{2}at^2\,, \qquad v^2 = u^2 + 2as$$

FURTHER MECHANICS

Motion of a projectile

Equation of trajectory is:

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

Elastic strings and springs

$$T = \frac{\lambda x}{l}\,, \qquad E = \frac{\lambda x^2}{2l}$$

Motion in a circle

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$$\omega^2 r \qquad \text{or} \qquad \frac{v^2}{r}$$

Centres of mass of uniform bodies

Triangular lamina: $\tfrac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\tfrac{3}{8}r$ from centre

Hemispherical shell of radius r: $\tfrac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\dfrac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\dfrac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\tfrac{3}{4}h$ from vertex

PROBABILITY & STATISTICS

Summary statistics

For ungrouped data:

$$\overline{x} = \frac{\Sigma x}{n}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\overline{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \overline{x}^2}$$

For grouped data:

$$\overline{x} = \frac{\Sigma xf}{\Sigma f}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\overline{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \overline{x}^2}$$

Discrete random variables

$$\mathrm{E}(X) = \Sigma xp, \qquad \mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2$$

For the binomial distribution $\mathrm{B}(n, p)$:

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \qquad \mu = np, \qquad \sigma^2 = np(1-p)$$

For the geometric distribution Geo(p):

$$p_r = p(1-p)^{r-1}, \qquad \mu = \frac{1}{p}$$

For the Poisson distribution $\mathrm{Po}(\lambda)$

$$p_r = \mathrm{e}^{-\lambda} \frac{\lambda^r}{r!}, \qquad \mu = \lambda, \qquad \sigma^2 = \lambda$$

Continuous random variables

$$\mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{Var}(X) = \int x^2\,\mathrm{f}(x)\,\mathrm{d}x - \{\mathrm{E}(X)\}^2$$

Sampling and testing

Unbiased estimators:

$$\overline{x} = \frac{\Sigma x}{n}, \qquad s^2 = \frac{\Sigma(x-\overline{x})^2}{n-1} = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Central Limit Theorem:

$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$\mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$$

FURTHER PROBABILITY & STATISTICS

Sampling and testing

Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \overline{x}_1)^2 + \Sigma(x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$$

Probability generating functions

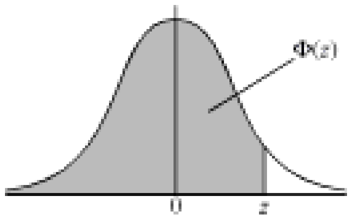
$\mathrm{G}_X(t) = \mathrm{E}(t^X)$, $\mathrm{E}(X) = \mathrm{G}'_X(1)$, $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \{\mathrm{G}'_X(1)\}^2$

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1, then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = P(Z \leqslant z).$$

For negative values of z, use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1, then, for each value of p, the table gives the value of z such that

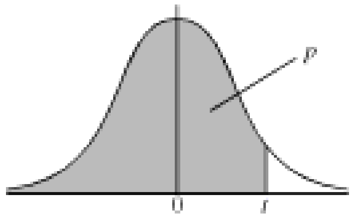
$$P(Z \leqslant z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE *t*-DISTRIBUTION

If T has a t-distribution with ν degrees of freedom, then, for each pair of values of p and ν, the table gives the value of t such that:

$P(T \leqslant t) = p$.

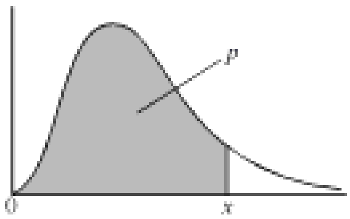


p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
$\nu = 1$	1.000	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.894	6.869
6	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.689
28	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.660
30	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE χ^2 -DISTRIBUTION

If X has a χ^2 -distribution with ν degrees of freedom then, for each pair of values of p and ν, the table gives the value of x such that

$$P(X \leqslant x) = p.$$



p	0.01	0.025	0.05	0.9	0.95	0.975	0.99	0.995	0.999
$\nu = 1$	$0.0^{3}1571$	$0.0^{3}9821$	$0.0^{2}3932$	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

WILCOXON SIGNED-RANK TEST

The sample has size n.

P is the sum of the ranks corresponding to the positive differences.

Q is the sum of the ranks corresponding to the negative differences.

T is the smaller of P and Q.

For each value of n the table gives the **largest** value of T which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *T*

	Level of significance			
One-tailed	0.05	0.025	0.01	0.005
Two-tailed	0.1	0.05	0.02	0.01
$n = 6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

For larger values of n, each of P and Q can be approximated by the normal distribution with mean $\frac{1}{4}n(n+1)$ and variance $\frac{1}{24}n(n+1)(2n+1)$.

WILCOXON RANK-SUM TEST

The two samples have sizes m and n, where $m \leqslant n$.

R_m is the sum of the ranks of the items in the sample of size m.

W is the smaller of R_m and $m(n + m + 1) - R_m$.

For each pair of values of m and n, the table gives the **largest** value of W which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *W*

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 3$			$m = 4$			$m = 5$			$m = 6$		
3	6	–	–									
4	6	–	–	11	10	–						
5	7	6	–	12	11	10	19	17	16			
6	8	7	–	13	12	11	20	18	17	28	26	24
7	8	7	6	14	13	11	21	20	18	29	27	25
8	9	8	6	15	14	12	23	21	19	31	29	27
9	10	8	7	16	14	13	24	22	20	33	31	28
10	10	9	7	17	15	13	26	23	21	35	32	29

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 7$			$m = 8$			$m = 9$			$m = 10$		
7	39	36	34									
8	41	38	35	51	49	45						
9	43	40	37	54	51	47	66	62	59			
10	45	42	39	56	53	49	69	65	61	82	78	74

For larger values of m and n, the normal distribution with mean $\frac{1}{2}m(m+n+1)$ and variance $\frac{1}{12}mn(m+n+1)$ should be used as an approximation to the distribution of R_m.

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Syllabus 26-27 Further Pure Mathematics 2

2 Further Pure Mathematics 2 (for Paper 2)

Knowledge of Paper 1: Further Pure Mathematics 1 subject content from this syllabus is assumed for this component.

2.1 Hyperbolic functions

Candidates should be able to:	Notes and examples
• understand the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, $\coth x$ in terms of the exponential function	
• sketch the graphs of hyperbolic functions	
• prove and use identities involving hyperbolic functions	e.g. $\cosh^2 x - \sinh^2 x \equiv 1$, $\sinh 2x \equiv 2\sinh x \cosh x$, and similar results corresponding to the standard trigonometric identities.
• understand and use the definitions of the inverse hyperbolic functions and derive and use the logarithmic forms	

2.2 Matrices

Candidates should be able to:	Notes and examples
• formulate a problem involving the solution of 3 linear simultaneous equations in 3 unknowns as a problem involving the solution of a matrix equation, or vice versa	
• understand the cases that may arise concerning the consistency or inconsistency of 3 linear simultaneous equations, relate them to the singularity or otherwise of the corresponding matrix, solve consistent systems, and interpret geometrically in terms of lines and planes	e.g. three planes meeting in a common point, or in a common line, or having no common points.
• understand the terms 'characteristic equation', 'eigenvalue' and 'eigenvector', as applied to square matrices	Including use of the definition $\mathbf{Ae} = \lambda\mathbf{e}$ to prove simple properties, e.g. that λ^n is an eigenvalue of $\mathbf{A}^n$.
• find eigenvalues and eigenvectors of 2×2 and 3×3 matrices	Restricted to cases where the eigenvalues are real and distinct.
• express a square matrix in the form $\mathbf{QDQ}^{-1}$, where $\mathbf{D}$ is a diagonal matrix of eigenvalues and $\mathbf{Q}$ is a matrix whose columns are eigenvectors, and use this expression	e.g. in calculating powers of 2×2 or 3×3 matrices.
• use the fact that a square matrix satisfies its own characteristic equation.	e.g. in finding successive powers of a matrix or finding an inverse matrix; restricted to 2×2 or 3×3 matrices only.

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2.3 Differentiation

Candidates should be able to:	Notes and examples
• differentiate hyperbolic functions and differentiate $\sin^{-1}x$, $\cos^{-1}x$, $\sinh^{-1}x$, $\cosh^{-1}x$ and $\tanh^{-1}x$	
• obtain an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in cases where the relation between x and y is defined implicitly or parametrically	
• derive and use the first few terms of a Maclaurin's series for a function.	Derivation of a general term is not included, but successive 'implicit' differentiation steps may be required, e.g. for $y = \tan x$ following an initial differentiation rearranged as $y' = 1 + y^2$.

2.4 Integration

Candidates should be able to:	Notes and examples
• integrate hyperbolic functions and recognise integrals of functions of the form $\frac{1}{\sqrt{a^2 - x^2}}$, $\frac{1}{\sqrt{x^2 + a^2}}$ and $\frac{1}{\sqrt{x^2 - a^2}}$, and integrate associated functions using trigonometric or hyperbolic substitutions as appropriate	Including use of completing the square where necessary, e.g. to integrate $\frac{1}{\sqrt{x^2 + x}}$.
• derive and use reduction formulae for the evaluation of definite integrals	e.g. $\int_0^{\frac{1}{2}\pi} \sin^n x \,\mathrm{d}x$, $\int_0^1 \mathrm{e}^{-x}(1-x)^n \,\mathrm{d}x$. In harder cases hints may be given, e.g. $\int_0^{\frac{1}{4}\pi} \sec^n x \,\mathrm{d}x$ by considering $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x \sec^n x)$.
• understand how the area under a curve may be approximated by areas of rectangles, and use rectangles to estimate or set bounds for the area under a curve or to derive inequalities or limits concerning sums	Questions may involve either rectangles of unit width or rectangles whose width can tend to zero, e.g. $1 + \ln n > \sum_{r=1}^{n} \frac{1}{r} > \ln(n+1)$, $\sum_{r=1}^{n} \frac{1}{n}\left(1 + \frac{r}{n}\right)^{-1} \approx \int_0^1 (1+x)^{-1} \,\mathrm{d}x$.

continued

CHAPTER 8. DIFFERENTIAL EQUATIONS

2 Further Pure Mathematics 2

2.4 Integration continued

Candidates should be able to:	Notes and examples
• use integration to find – arc lengths for curves with equations in Cartesian coordinates, including the use of a parameter, or in polar coordinates – surface areas of revolution about one of the axes for curves with equations in Cartesian coordinates, including the use of a parameter.	Any questions involving integration may require techniques from Cambridge International A Level Mathematics (9709) applied to more difficult cases, e.g. integration by parts for $\int e^x \sin x \, dx$, or use of the substitution $t = \tan\frac{1}{2}x$. Surface areas of revolution for curves with equations in polar coordinates will not be required.

2.5 Complex numbers

Candidates should be able to:	Notes and examples
• understand de Moivre's theorem, for a positive or negative integer exponent, in terms of the geometrical effect of multiplication and division of complex numbers	
• prove de Moivre's theorem for a positive integer exponent	e.g. by induction.
• use de Moivre's theorem for a positive or negative rational exponent – to express trigonometrical ratios of multiple angles in terms of powers of trigonometrical ratios of the fundamental angle – to express powers of $\sin\theta$ and $\cos\theta$ in terms of multiple angles – in the summation of series – in finding and using the nth roots of unity.	e.g. expressing $\cos 5\theta$ in terms of $\cos\theta$ or $\tan 5\theta$ in terms of $\tan\theta$. e.g. expressing $\sin^6\theta$ in terms of $\cos 2\theta$, $\cos 4\theta$ and $\cos 6\theta$. e.g. using the '$C + iS$' method to sum series such as $\sum_{r=1}^{n} \binom{n}{r} \sin r\theta$.

2.6 Differential equations

Candidates should be able to:	Notes and examples
• find an integrating factor for a first order linear differential equation, and use an integrating factor to find the general solution	e.g. $\frac{dy}{dx} - 2y = x^2$, $x\frac{dy}{dx} - y = x^4$, $\frac{dy}{dx} + y\coth x = \cosh x$.
• recall the meaning of the terms 'complementary function' and 'particular integral' in the context of linear differential equations, and recall that the general solution is the sum of the complementary function and a particular integral	

continued

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2.6 Differential equations continued

Candidates should be able to:	Notes and examples
• find the complementary function for a first or second order linear differential equation with constant coefficients	For second order equations, including the cases where the auxiliary equation has distinct real roots, a repeated real root or conjugate complex roots.
• recall the form of, and find, a particular integral for a first or second order linear differential equation in the cases where a polynomial or $a\,\mathrm{e}bx$ or $a\cos px + b\sin px$ is a suitable form, and in other simple cases find the appropriate coefficient(s) given a suitable form of particular integral	e.g. evaluate k given that $kx\cos 2x$ is a particular integral of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin 2x$.
• use a given substitution to reduce a differential equation to a first or second order linear equation with constant coefficients or to a first order equation with separable variables	e.g. the substitution $x = \mathrm{e}t$ to reduce to linear form a differential equation with terms of the form $ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy$, or the substitution $y = ux$ to reduce $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$ to separable form.
• use initial conditions to find a particular solution to a differential equation, and interpret a solution in terms of a problem modelled by a differential equation.	