The Question Bank of

Further Pure Mathematics 1

for CAIE 9231 paper 1.

v1.0

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Instructions for Use

- This question bank is organized by chapter for systematic revision.
- This question bank is complied based on the 26-27 CAIE Further Pure Mathematics 1 syllabus, which is included as appendix.
- Each question includes its source for reference.
- Mark schemes are provided in the separate answer booklet.
- The formula sheet (MF19) is included as appendix.
- Use this resource for targeted practice and exam preparation.

Contents

Chapter 1

Roots of polynomial equations

1. [9231/s25/11/q2]

The cubic equation $x^3 + 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3 - 1,\ \beta^3 - 1,\ \gamma^3 - 1$. [3]

(b) Find the value of $\left(\alpha^3 - 1\right)^2 + \left(\beta^3 - 1\right)^2 + \left(\gamma^3 - 1\right)^2$. [2]

(c) Find the value of $\left(\alpha^3 - 1\right)^3 + \left(\beta^3 - 1\right)^3 + \left(\gamma^3 - 1\right)^3$. [2]

2. [9231/s25/13/q3]

The quartic equation $x^4 + 7x^2 + 3x + 22 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [2]

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

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(c) Use standard results from the list of formulae (MF19) to find the value of

$$\sum_{r=1}^{10}\left((\alpha^2+r)^2+(\beta^2+r)^2+(\gamma^2+r)^2+(\delta^2+r)^2\right).$$ [5]

3. [9231/s25/14/q4]

The cubic equation $x^3 + bx^2 + cx - 1 = 0$, where b and c are constants, has roots α, β, γ.

It is given that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular.

(a) Show that $\alpha^2 + \beta^2 + \gamma^2 = 3$. [4]

(b) It is given that $\alpha^3 + \beta^3 + \gamma^3 = 3$ and that the constants b and c are positive.

Find the values of b and c. [6]

4. [9231/w25/11/q4]

The quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Show that a quartic equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1, 2\delta + 1$ is

$$y^4 - 2y^3 + 4y^2 + 2y + 11 = 0.$$ [4]

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n + (2\delta + 1)^n$ is denoted by S_n.

(b) Find the value of S_2. [2]

(c) Given that $S_3 = -22$, find the value of S_4. [2]

5. [9231/w25/12/q2]

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β and γ. It is given that

$$\alpha + \beta + \gamma = 2,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 3,$$
$$\alpha^4 + \beta^4 + \gamma^4 = 5.$$

(a) Find the values of b and c. [3]

(b) Find the value of d. [5]

6. [9231/w25/14/q2]

The cubic equation $x^3 + bx^2 + cx + d = 0$, where $d \neq 0$, has roots α, β, γ such that $\gamma = \dfrac{1}{\beta}$.

(a) Show that $\beta + \dfrac{1}{\beta} = d - b$. [3]

(b) Show also that $\beta + \dfrac{1}{\beta} = \dfrac{1-c}{d}$. [2]

(c) It is given that $b = 3$, $c = -3$ and $d > 0$.

(i) Find the value of d. [2]

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

7. [9231/s24/11/q1]

The cubic equation $2x^3 + x^2 - px - 5 = 0$, where p is a positive constant, has roots α, β, γ.

(a) State, in terms of p, the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [1]

(b) Find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. [2]

(c) Deduce a cubic equation whose roots are $\alpha\beta$, $\beta\gamma$, $\alpha\gamma$. [1]

(d) Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{3}$, find the value of p. [2]

8. [9231/s24/13/q2]

The cubic equation $x^3 + 2x^2 + 3x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2 + 1, \beta^2 + 1, \gamma^2 + 1$. [3]

(b) Find the value of $(\alpha^2 + 1)^2 + (\beta^2 + 1)^2 + (\gamma^2 + 1)^2$. [2]

(c) Find the value of $(\alpha^2 + 1)^3 + (\beta^2 + 1)^3 + (\gamma^2 + 1)^3$. [2]

9. [9231/w24/11/q3]

The quartic equation $x^4 + 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^4, \beta^4, \gamma^4, \delta^4$ and state the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [5]

(b) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [3]

(c) Find the value of $\alpha^8 + \beta^8 + \gamma^8 + \delta^8$. [2]

10. [9231/w24/12/q3]

It is given that

$$\begin{aligned}\alpha + \beta + \gamma + \delta &= 2,\\ \alpha^2+\beta^2+\gamma^2+\delta^2 &= 3,\\ \alpha^3+\beta^3+\gamma^3+\delta^3 &= 4.\end{aligned}$$

(a) Find the value of $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$. [2]

(b) Find the value of $\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\beta^2\alpha+\beta^2\gamma+\beta^2\delta+\gamma^2\alpha+\gamma^2\beta+\gamma^2\delta+\delta^2\alpha+\delta^2\beta+\delta^2\gamma$. [3]

(c) It is given that α, β, γ, δ are the roots of the equation

$$6x^4-12x^3+3x^2+2x+6=0.$$

(i) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [3]

(ii) Find the value of $\alpha^5+\beta^5+\gamma^5+\delta^5$. [2]

11. [9231/s23/11/q2]

The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2 + (\beta+r)^2 + (\gamma+r)^2\right) = n(n^2 + an + b),$$

where a and b are constants to be determined. [6]

12. [9231/s23/13/q3]

The equation $x^4 - x^2 + 2x + 5 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [4]

(b) Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

13. [9231/w23/11/q3]

The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots α, β, γ, δ. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

14. [9231/w23/12/q4]

The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1$, $3\beta + 1$, $3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$ [3]

The sum $(3\alpha + 1)^n + (3\beta + 1)^n + (3\gamma + 1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

(c) Find the values of S_{-1} and S_{-2}. [3]

15. [9231/s22/11/q4]

The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9} + \frac{1}{\beta^9} + \frac{1}{\gamma^9}$. [2]

16. **[9231/s22/13/q2]**

The cubic equation $x^3 + 5x^2 + 10x - 2 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(b) Show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

17. [9231/w22/11/q1]

The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

18. [9231/w22/12/q2]

The equation $x^4 + 3x^2 + 2x + 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [4]

(b) Find the value of $\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$. [2]

19. [9231/s21/11/q3]

The equation $x^4 - 2x^3 - 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^3, β^3, γ^3, δ^3 and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

20. **[9231/s21/13/q2]**

The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. [3]

(d) Find the value of $\dfrac{1}{\alpha+\beta} + \dfrac{1}{\beta+\gamma} + \dfrac{1}{\gamma+\alpha}$. [2]

21. [9231/w21/11/q1]

It is given that

$$\alpha+\beta+\gamma=3, \quad \alpha^2+\beta^2+\gamma^2=5, \quad \alpha^3+\beta^3+\gamma^3=6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

22. [9231/w21/12/q4]

The cubic equation $x^3+2x^2+3x+3=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [2]

(b) Show that $\alpha^3+\beta^3+\gamma^3=1$. [2]

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

23. [9231/s20/11/q2]

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

24. [9231/s20/13/q1]

The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

25. [9231/w20/11/q3]

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

26. [9231/w20/12/q1]

The cubic equation $x^3+bx^2+cx+d=0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma=-1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha+1$, $\beta+1$, $\gamma+1$. [3]

(c) Given also that $\gamma+1=-\alpha-1$, deduce that $(c-2b+3)(b-3)=b-c$. [4]

27. [9231/s19/11/q6]

The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

28. [9231/s19/13/q9]

A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

(iii) Show that $b^3 - 15b + 126 = 0$. [4]

(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

29. [9231/w19/11/q7]

The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

(ii) Show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$. [3]

(iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$. [2]

30. [9231/s18/11/q4]

It is given that the equation

$$x^3 - 21x^2 + kx - 216 = 0,$$

where k is a constant, has real roots a, ar and ar^{-1}.

(i) Find the numerical values of the roots. [6]

(ii) Deduce the value of k. [2]

31. [9231/s18/13/q6]

The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$ [2]

The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

(iii) Find the value of S_{-2}. [4]

32. [9231/w18/11/q2]

The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are α, 2α, 4α, where p, q, r and α are non-zero real constants.

(i) Show that

$$2p\alpha + q = 0. \qquad [4]$$

(ii) Show that

$$p^3 r - q^3 = 0. \qquad [2]$$

33. [9231/w18/12/q1]

The roots of the cubic equation

$$x^3 - 5x^2 + 13x - 4 = 0$$

are α, β, γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

34. [9231/s17/11/q7]

By finding a cubic equation whose roots are α, β and γ, solve the set of simultaneous equations

$$\begin{aligned} \alpha + \beta + \gamma &= -1, \\ \alpha^2 + \beta^2 + \gamma^2 &= 29, \\ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= -1. \end{aligned}$$ [8]

35. [9231/s17/13/q1]

The roots of the cubic equation $x^3 + 2x^2 - 3 = 0$ are α, β and γ.

(i) By using the substitution $y = \dfrac{1}{x^2}$, find the cubic equation with roots $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$. [3]

(ii) Hence find the value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. [1]

(iii) Find also the value of $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\gamma^2\alpha^2}$. [1]

36. [9231/w17/11/q4]

The cubic equation $2x^3 - 3x^2 + 4x - 10 = 0$ has roots α, β and γ.

(i) Find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. [4]

(ii) Find the value of $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$. [4]

37. [9231/s16/11/q1]

The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \frac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \frac{1}{\alpha}$, $1 + \frac{1}{\beta}$ and $1 + \frac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

38. [9231/s16/13/q8]

The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Show that the cubic equation with roots $\frac{\alpha - 1}{\alpha}$, $\frac{\beta - 1}{\beta}$ and $\frac{\gamma - 1}{\gamma}$ may be found using the substitution $z = \frac{1}{1 - x}$, and find this equation, giving your answer in the form $px^3 + qx^2 + rx + s = 0$, where p, q, r and s are constants to be determined. [4]

39. [9231/w16/11/q2]

Find the cubic equation with roots α, β and γ such that

$$\alpha + \beta + \gamma = 3,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha^3 + \beta^3 + \gamma^3 = -30,$$

giving your answer in the form $x^3 + px^2 + qx + r = 0$, where p, q and r are integers to be found. [6]

40. [9231/s15/11/q4]

The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β and γ. Find the values of

(i) $\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}$,

(ii) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$,

(iii) $\dfrac{1}{\alpha^2\beta\gamma} + \dfrac{1}{\alpha\beta^2\gamma} + \dfrac{1}{\alpha\beta\gamma^2}$.

[6]

Deduce a cubic equation, with integer coefficients, having roots $\dfrac{1}{\alpha\beta}$, $\dfrac{1}{\beta\gamma}$ and $\dfrac{1}{\gamma\alpha}$. [2]

41. [9231/s15/13/q1]

The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

42. [9231/w15/11/q5]

The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

Chapter 2

Rational functions and graphs

1. [9231/s25/11/q7]

The curve C has equation $y = \dfrac{2x^2 - 5x}{2x^2 - 7x - 4}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{2x^2 - 5x}{2x^2 - 7x - 4}\right|$. [1]

(e) Find in exact form the set of values of x for which $\left|\dfrac{2x^2 - 5x}{2x^2 - 7x - 4}\right| < \dfrac{1}{9}$. [5]

2. [9231/s25/13/q6]

The curve C has equation $y = \frac{x^2+a}{x+a}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

(b) Find, in terms of a, the x-coordinates of the stationary points on C. [3]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{x^2+a}{x+a}\right|$. [1]

(e) Find the set of values of a for which $\left|\frac{x^2+a}{x+a}\right| = a$ has two real solutions. [4]

3. [9231/s25/14/q7]

The curve C has equation $y = \dfrac{x^2+x-4}{x^2+x+2}$.

(a) State the equation of the asymptote of C. [1]

(b) Show that, for all real values of x, $-\frac{17}{7} \leqslant y < 1$. [4]

(c) Find the coordinates of any stationary points of C. [3]

(d) Sketch C, stating the coordinates of the intersections with the axes. [3]

(e) Sketch the graph with equation $y = \dfrac{|x|^2+|x|-4}{|x|^2+|x|+2}$ and find the set of values of x for which

$\dfrac{|x|^2+|x|-4}{|x|^2+|x|+2} < -\frac{1}{2}$. [5]

4. [9231/w25/11/q7]

The curve C has equation $y = \dfrac{x+2}{x^2+3x+1}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Show that C has no stationary points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x+2}{x^2+3x+1}\right|$. [2]

(e) Find in exact form the set of values of x for which $\left|\dfrac{x+2}{x^2+3x+1}\right| > 2$. [4]

5. [9231/w25/12/q7]

The curve C has equation $y = \dfrac{x^2 + x + 1}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of any stationary points on C. [3]

(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \dfrac{|x|^2 + |x| + 1}{|x| + 1}$. [2]

(e) Find, in exact form, the set of values of x for which $\dfrac{|x|^2 + |x| + 1}{|x| + 1} < 3$. [3]

6. [9231/w25/14/q7]

The curve C has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no stationary points. [4]

(c) Sketch C, stating the coordinates of the points of intersection with the axes. [3]

....

(d) Sketch the curve with equation $y = \left|\dfrac{10x^2 - 11x - 18}{10x - 18}\right|$. [2]

(e) Given that $\left|\dfrac{10x^2 - 11x - 18}{10x - 18}\right| < 4$ for $\dfrac{-29 - \sqrt{4441}}{20} < x < p$ and $\dfrac{-29 + \sqrt{4441}}{20} < x < q$, find the values of p and q. [4]

7. [9231/s24/11/q6]

The curve C has equation $y = \dfrac{x^2 + ax + 1}{x + 2}$, where $a > \frac{5}{2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no stationary points. [4]

(c) Sketch C, stating the coordinates of the point of intersection with the y-axis and labelling the asymptotes. [3]

(d) **(i)** Sketch the curve with equation $y = \left|\dfrac{x^2 + ax + 1}{x + 2}\right|$. [2]

(ii) On your sketch in part **(i)**, draw the line $y = a$. [1]

(iii) It is given that $\left|\dfrac{x^2 + ax + 1}{x + 2}\right| < a$ for $-5 - \sqrt{14} < x < -3$ and $-5 + \sqrt{14} < x < 3$.

Find the value of a. [2]

8. [9231/s24/13/q6]

The curve C has equation $y = \dfrac{x+1}{x^2+3}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch $y^2 = \dfrac{x+1}{x^2+3}$, stating the coordinates of the stationary points and the intersections with the axes. [4]

9. [9231/w24/11/q6]

The curve C has equation $y = \dfrac{4x^2 + x + 1}{2x^2 - 7x + 3}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [5]

(d) Sketch the curve with equation $y = \left|\dfrac{4x^2 + x + 1}{2x^2 - 7x + 3}\right|$ and state the set of values of k for which $\left|\dfrac{4x^2 + x + 1}{2x^2 - 7x + 3}\right| = k$ has 4 distinct real solutions. [2]

10. [9231/w24/12/q6]

The curve C has equation $y = \frac{x^2+3}{x^2+1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]

(b) Show that $1 < y \leqslant 3$ for all real values of x. [4]

(c) Find the coordinates of any stationary points on C. [2]

(d) Sketch C, stating the coordinates of any intersections with the axes and labelling the asymptote. [3]

(e) Sketch the curve with equation $y = \frac{x^2+1}{x^2+3}$ and find the set of values of x for which $\frac{x^2+1}{x^2+3} < \frac{1}{2}$. [4]

11. [9231/s23/11/q6]

The curve C has equation $y = \frac{x^2+2x-15}{x-2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no stationary points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{x^2+2x-15}{x-2}\right|$. [2]

(e) Find the set of values of x for which $\left|\frac{2x^2+4x-30}{x-2}\right| < 15$. [4]

12. [9231/s23/13/q7]

The curve C has equation $y = \dfrac{x^2+2x+1}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the turning points on C. [3]

(c) Sketch C. [3]

(d) Sketch the curves with equations $y = \left|\dfrac{x^2+2x+1}{x-3}\right|$ and $y^2 = \dfrac{x^2+2x+1}{x-3}$ on a single diagram, clearly identifying each curve. [4]

13. [9231/w23/11/q7]

The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2+2}{x^2-x-2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C, giving your answers correct to 1 decimal place. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

(e) Find the set of values for which $\dfrac{1}{\mathrm{f}(x)} < \mathrm{f}(x)$. [4]

14. [9231/w23/12/q7]

The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of any stationary points on C. [2]

(c) Sketch C. [3]

(d) Find the coordinates of any stationary points on the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

(e) Sketch the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$ and find, in exact form, the set of values for which $\dfrac{1}{\mathrm{f}(x)} > \mathrm{f}(x)$. [6]

15. **[9231/s22/11/q5]**

The curve C has equation $y=\dfrac{2x^2-x-1}{x^2+x+1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y=\left|\dfrac{2x^2-x-1}{x^2+x+1}\right|$ and state the set of values of k for which $\left|\dfrac{2x^2-x-1}{x^2+x+1}\right|=k$ has 4 distinct real solutions. [2]

16. [9231/s22/13/q1]

(a) Sketch the curve with equation $y = \dfrac{x+1}{x-1}$. [2]

(b) Sketch the curve with equation $y = \dfrac{|x|+1}{|x|-1}$ and find the set of values of x for which $\dfrac{|x|+1}{|x|-1} < -2$. [4]

17. [9231/s22/13/q3]

A curve C has equation $y = \dfrac{ax^2 + x - 1}{x - 1}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 1 + 4a$. [4]

(c) Sketch C. You do not need to find the coordinates of the intersections with the axes. [3]

18. [9231/w22/11/q7]

The curve C has equation $y = \dfrac{5x^2}{5x-2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{5x^2}{5x-2}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{5x^2}{5x-2}\right| < 2$. [6]

19. [9231/w22/12/q7]

The curve C has equation $y = \dfrac{x^2 - x}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the exact coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{x^2 - x}{x+1}\right| < 6$. [5]

20. [9231/s21/11/q7]

The curve C has equation $y = \dfrac{x^2+x+9}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2|x^2+x+9| > 13|x+1|$. [5]

21. [9231/s21/13/q7]

The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left| \dfrac{x^2 - x - 3}{1 + x - x^2} \right|$ and find in exact form the set of values of x for which $\left| \dfrac{x^2 - x - 3}{1 + x - x^2} \right| < 3$. [6]

22. [9231/w21/11/q7]

The curve C has equation $y = \dfrac{4x+5}{4-4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{4x+5}{4-4x^2}\right|$ and find in exact form the set of values of x for which $4|4x+5| > 5|4-4x^2|$. [6]

23. [9231/w21/12/q6]

The curve C has equation $y = \dfrac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $0 < y < 12$. [4]

(c) Sketch C. [2]

(d) **(i)** Sketch the graphs of $y = \left|\dfrac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(ii) Use your sketch to find the set of values of c for which $\left|\dfrac{x^2}{x-3}\right| \leqslant |x| + c$ has no solution. [1]

24. [9231/s20/11/q1]

Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$.

[4]

25. [9231/s20/11/q3]

The curve C has equation $y = \dfrac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

(c) Sketch C. [3]

26. [9231/s20/13/q6]

The curve C has equation $y = \dfrac{10+x-2x^2}{2x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{10+x-2x^2}{2x-3}\right| < 4$. [6]

27. [9231/w20/11/q6]

The curve C has equation $y = \dfrac{x^2+x-1}{x-1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x-1}{x-1}\right|$. [2]

28. [9231/w20/12/q6]

Let a be a positive constant.

(a) The curve C_1 has equation $y = \dfrac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\dfrac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\dfrac{x-a}{x-2a}\right|$.

(b) **(i)** Find the coordinates of any stationary points of C_2. [3]

(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\dfrac{x-a}{x-2a}\right)^2 \leqslant \left|\dfrac{x-a}{x-2a}\right|$. [5]

29. [9231/s19/11/q10]

The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

(ii) Find the equation of the oblique asymptote of C_2. [2]

(iii) Show that C_1 and C_2 do not intersect. [2]

(iv) Find the coordinates of the stationary points of C_2. [3]

(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

30. [9231/s19/13/q6]

The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the stationary points of C. [3]

(iii) Sketch C. [3]

31. [9231/w19/11/q4]

The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

(ii) State the equation of the other asymptote of C. [1]

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

32. [9231/s18/11/q6]

The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C does not intersect the x-axis. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [3]

33. [9231/s18/13/q4]

The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

(i) Find the coordinates of the points of intersection of C with the axes. [2]

(ii) Find the equation of each of the asymptotes of C. [3]

(iii) Sketch C. [3]

34. [9231/w18/11/q6]

The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C intersects the x-axis twice. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C, stating the coordinates of its point of intersection with the y-axis. [3]

35. [9231/w18/12/q9]

The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

(i) Find the equation of the asymptote of C. [2]

(ii) Show that, for all real values of x, $-\frac{1}{3} \leqslant y < 5$. [4]

(iii) Find the coordinates of any stationary points of C. [2]

(iv) Sketch C, stating the coordinates of any intersections with the y-axis. [2]

36. [9231/s17/11/q9]

The curve C has equation $y = \dfrac{x^2 - 3x + 6}{1 - x}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the turning points of C. [3]

(iii) Find the coordinates of any intersections with the coordinate axes. [2]

(iv) Sketch C. [3]

37. [9231/w17/11/q9]

The curve C has equation

$$y = \frac{3x - 9}{(x - 2)(x + 1)}.$$

(i) Find the equations of the asymptotes of C. [2]

(ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$. [4]

(iii) Find the coordinates of the turning points of C. [3]

(iv) Sketch C. [3]

38. [9231/s16/11/q7]

A curve C has equation $y = \frac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $0 < y < 8$. [4]

Sketch C, giving the coordinates of the turning points. [3]

39. [9231/s16/13/q5]

The curve C has equation $y = \dfrac{x+2}{x^2-9}$. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ at all points on C. [3]

State the equations of the asymptotes of C. [2]

Sketch C, showing the coordinates of any points of intersection with the coordinate axes. [3]

40. [9231/s15/13/q10]

The curve C has equation $y = \dfrac{4x^2 - 3x}{x^2 + 1}$. Verify that the equation of C may be written in the form $y = -\dfrac{1}{2} + \dfrac{(3x-1)^2}{2(x^2+1)}$ and also in the form $y = \dfrac{9}{2} - \dfrac{(x+3)^2}{2(x^2+1)}$. [3]

Hence show that $-\frac{1}{2} \leqslant y \leqslant \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of C. [2]

State the equation of the asymptote of C. [1]

Sketch the graph of C, stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

41. [9231/w15/11/q8]

The curve C has equation $y = \frac{2x^2 + kx}{x+1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes. [6]

Chapter 3

Summation of series

1. [9231/s25/11/q1]

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(2-3r)(5-3r) = an^3+bn^2+cn,$$

where a, b and c are integers to be determined. [3]

(b) Use the method of differences to find $\sum_{r=1}^{n}\frac{1}{(2-3r)(5-3r)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty}\frac{1}{(2-3r)(5-3r)}$. [1]

2. [9231/s25/13/q4]

Let $w_r = r(r+1)(r+2)\dots(r+9)$.

(a) Show that

$$w_{r+1} - w_r = 10(r+1)(r+2)\dots(r+9).$$ [2]

....

(b) Given that $u_r = (r+1)(r+2)\dots(r+9)$, find $\sum_{r=1}^{n} u_r$ in terms of n. [3]

(c) Given that $v_r = x^{w_{r+1}} - x^{w_r}$, find the set of values of x for which the infinite series

$$v_1 + v_2 + v_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

3. [9231/s25/14/q1]

(a) Use the List of formulae (MF19) to find $\sum_{r=1}^{n}(2r+1)$ in terms of n, simplifying your answer. [2]

(b) Show that $\dfrac{2r+1}{(r^2+1)(r^2+2r+2)} = \dfrac{1}{r^2+1} - \dfrac{1}{r^2+2r+2}$. [1]

(c) Use the method of differences to find $\sum_{r=1}^{n}\dfrac{2r+1}{(r^2+1)(r^2+2r+2)}$. [2]

(d) Deduce the value of $\sum_{r=1}^{\infty}\dfrac{2r+1}{(r^2+1)(r^2+2r+2)}$. [1]

4. [9231/w25/11/q1]

(a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}\left(8r^3+12r^2+4r+3\right)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1}{r^4}-\frac{1}{(r+1)^4}=\frac{4r^3+6r^2+4r+1}{r^4(r+1)^4}$$

and hence use the method of differences to find $\sum_{r=1}^{n}\frac{4r^3+6r^2+\quad+1}{r^4(r+1)^4}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty}\frac{4r^3+6r^2+4r+1}{r^4(r+1)^4}$. [1]

5. [9231/w25/12/q1]

(a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(r^3 - r)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{r+3}{r^3-r}$ in the form $\frac{A}{r-1} + \frac{B}{r} + \frac{C}{r+1}$, where A, B and C are constants to be determined, and hence use the method of differences to find $\sum_{r=2}^{n}\frac{r+3}{r^3-r}$. [5]

(c) Deduce the value of $\sum_{r=2}^{\infty}\frac{r+3}{r^3-r}$. [1]

6. [9231/w25/14/q3]

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(r+1)(r+2)(r+3) = \tfrac{1}{4}n\left(n^3 + bn^2 + cn + d\right),$$

where b, c and d are integers to be determined. [3]

(b) Express $\dfrac{2}{(r+1)(r+2)(r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+2)(r+3)}.$$ [5]

(c) State the value of $\displaystyle\sum_{r=1}^{\infty}\frac{2}{(r+1)(r+2)(r+3)}$. [1]

7. [9231/s24/11/q3]

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{N} r(r+1)(3r+4) = \tfrac{1}{12}N(N+1)(N+2)(9N+19).$$ [3]

(b) Express $\dfrac{3r+4}{r(r+1)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{N} \frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}$$

in terms of N. [4]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{3r+4}{r(r+1)}\left(\frac{1}{4}\right)^{r+1}$. [1]

8. [9231/s24/13/q4]

(a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1). \qquad [5]$$

The sum S_n is defined by $S_n = \sum_{r=1}^{n} r^4$.

(b) Using the identity

$$(2r+1)^5 - (2r-1)^5 \equiv 160r^4 + 80r^2 + 2,$$

show that $S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$. [6]

(c) Find the value of $\lim_{n\to\infty}\left(n^{-5}S_n\right)$. [2]

9. [9231/w24/11/q4]

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{5k}{(5r+k)(5r+5+k)}$ in terms of n and k, where k is a positive constant. [4]

It is given that $\sum_{r=1}^{\infty} \frac{5k}{(5r+k)(5r+5+k)} = \frac{1}{3}$.

(b) Find the value of k. [2]

(c) Hence find $\sum_{r=n}^{n^2} \frac{5k}{(5r+k)(5r+5+k)}$ in terms of n. [2]

10. [9231/w24/12/q5]

It is given that $S_n = \sum_{r=1}^{n} u_r$, where $u_r = x^{\mathrm{f}(r)} - x^{\mathrm{f}(r+1)}$ and $x > 0$.

(a) Find S_n in terms of n, x and the function f. [2]

(b) Given that $\mathrm{f}(r) = \ln r$, find the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Given instead that $\mathrm{f}(r) = 2\log_x r$ where $x \neq 1$, use standard results from the List of formulae (MF19) to find $\sum_{n=1}^{N} S_n$ in terms of N. Fully factorise your answer. [4]

11. [9231/s23/11/q3]

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k , where k is a positive constant. [4]

(b) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(kr+1)(kr-k+1)}$. [1]

(c) Find also $\sum_{r=n}^{n^2} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k. [2]

12. [9231/s23/13/q2]

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(6r^2+6r-5) = an^3+bn^2+cn,$$

where a, b and c are integers to be determined. [2]

(b) Use the method of differences to find $\displaystyle\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}$ in terms of n. [4]

13. [9231/w23/11/q1]

(a) By considering $(r+1)^2 - r^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$ [4]

(b) Given that $\sum_{r=1}^{n}(r+a) = n$, find a in terms of n. [3]

14. [9231/w23/12/q1]

(a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(3r^2+3r+1)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1}{r^3}-\frac{1}{(r+1)^3}=\frac{3r^2+3r+1}{r^3(r+1)^3}$$

and hence use the method of differences to find $\sum_{r=1}^{n}\frac{3r^2+3r+1}{r^3(r+1)^3}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty}\frac{3r^2+3r+1}{r^3(r+1)^3}$. [1]

15. [9231/s22/11/q1]

Let a be a positive constant.

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

(b) Find the value of a for which $\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

16. [9231/s22/13/q4]

Let $u_r = \mathrm{e}^{rx}(\mathrm{e}^{2x} - 2\mathrm{e}^{x} + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

17. [9231/w22/11/q3]

(a) By considering $(2r+1)^3-(2r-1)^3$, use the method of differences to prove that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$ [5]

Let $S_n = 1^2+3\times2^2+3^2+3\times4^2+5^2+3\times6^2+\ldots+\left(2+(-1)^n\right)n^2$.

(b) Show that $S_{2n}=\frac{1}{3}n(2n+1)(an+b)$, where a and b are integers to be determined. [3]

(c) State the value of $\lim_{n\to\infty}\frac{S_{2n}}{n^3}$. [1]

18. [9231/w22/12/q1]

(a) Use the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+2)$ in terms of n, simplifying your answer. [2]

(b) Express $\frac{1}{r(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. [1]

19. [9231/s21/11/q2]

(a) Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\frac{r+1}{\left(r+1\right)^2+1}-\frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n}\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty}\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [1]

20. [9231/s21/13/q1]

(a) Show that

$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}. \qquad [2]$$

Let $u_r = \dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n} u_r$. [3]

(c) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

21. [9231/w21/11/q2]

(a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

22. [9231/w21/12/q3]

Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

23. [9231/s20/11/q4]

(a) By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$. [1]

(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2-1}$. [4]

24. [9231/s20/13/q3]

Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2 - 1)$. [4]

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

25. [9231/w20/11/q2]

(a) Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

(b) Use the method of differences to find $\displaystyle\sum_{r=1}^{n}\frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty}\frac{1}{(7r+1)(7r+8)}$. [1]

26. [9231/w20/12/q3]

(a) By simplifying $\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right)$, show that $\dfrac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$. [1]

Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \dfrac{1}{x^n - \sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

(c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

27. [9231/s19/11/q2]

Let $u_n = \dfrac{4\sin\left(n - \frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1) + \cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}.$$ [2]

(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

28. [9231/s19/13/q4]

(i) Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

29. [9231/w19/11/q5]

Let $S_N = \sum_{r=1}^{N} (5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N} \frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

(ii) Use the method of differences to express T_N in terms of N. [4]

(iii) Find $\lim_{N\to\infty} \left(N^{-3} S_N T_N\right)$. [2]

30. [9231/s18/11/q5]

Let $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$.

(i) Use the standard result for $\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n+1).$$ [4]

(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and find $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$. [4]

31. [9231/s18/13/q2]

(i) Verify that

$$\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}} = \frac{1}{n\mathrm{e}^n} - \frac{1}{(n+1)\mathrm{e}^{n+1}}.$$ [1]

Let $S_N = \sum_{n=1}^{N} \frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

Let $S = \lim_{N\to\infty} S_N$.

(iii) Find the least value of N such that $(N+1)(S - S_N) < 10^{-3}$. [3]

32. [9231/s18/13/q9]

For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \ldots + (u_r - 3)x^r + \ldots$$

is convergent. [2]

(iii) Use the result given in part **(i)** to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$ [3]

33. [9231/w18/11/q3]

The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

34. [9231/w18/11/q11e]

(i) By considering $(2r+1)^2 - (2r-1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1). \qquad [3]$$

(ii) By considering $(2r+1)^4 - (2r-1)^4$, use the method of differences and the result given in part **(i)** to prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2. \qquad [5]$$

The sums S and T are defined as follows:

$$S = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + (2N)^3 + (2N+1)^3,$$
$$T = 1^3 + 3^3 + 5^3 + 7^3 + \ldots + (2N-1)^3 + (2N+1)^3.$$

(iii) Use the result given in part **(ii)** to show that $S = (2N+1)^2(N+1)^2$. [1]

(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

(v) Deduce the value of $\dfrac{S}{T}$ as $N \to \infty$. [2]

35. [9231/w18/12/q7]

Let

$$S_N = \sum_{r=1}^{N} (3r+1)(3r+4) \quad \text{and} \quad T_N = \sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}.$$

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = N(3N^2 + 12N + 13).$$ [3]

(ii) Use the method of differences to show that

$$T_N = \frac{1}{12} - \frac{1}{3(3N+4)}.$$ [3]

(iii) Deduce that $\dfrac{S_N}{T_N}$ is an integer. [2]

(iv) Find $\lim_{N\to\infty} \dfrac{S_N}{N^3 T_N}$. [2]

36. [9231/s17/11/q1]

It is given that $\sum_{r=1}^{n} u_r = n^2(2n+3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

(ii) Find u_r. [3]

37. [9231/s17/13/q2]

(i) Verify that $\dfrac{2r+1}{r(r+1)(r+2)} = \dfrac{1}{2}\left\{\dfrac{(2r+1)(2r+3)}{(r+1)(r+2)} - \dfrac{(2r-1)(2r+1)}{r(r+1)}\right\}$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$. [2]

(iii) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$. [2]

38. [9231/w17/11/q1]

Find $\sum_{r=1}^{n}(4r-3)(4r+1)$, giving your answer in its simplest form. [4]

39. [9231/s16/11/q2]

Express $\dfrac{4}{r(r+1)(r+2)}$ in partial fractions and hence find $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$. [5]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$. [1]

40. [9231/s16/13/q1]

Verify that $\dfrac{1}{(3r+1)(3r+4)} = \dfrac{1}{3}\left(\dfrac{1}{3r+1} - \dfrac{1}{3r+4}\right)$. [1]

Let S_N denote $\displaystyle\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and let S denote $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$. Find the least value of N such that $S - S_N < \frac{1}{10\,000}$. [5]

41. [9231/w16/11/q1]

Use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1}$. [4]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

42. [9231/s15/11/q1]

Use the List of Formulae (MF10) to show that $\sum_{r=1}^{13}(3r^2 - 5r + 1)$ and $\sum_{r=0}^{9}(r^3 - 1)$ have the same numerical value. [4]

43. [9231/s15/11/q8]

By considering $\sum_{r=1}^{n} z^{2r-1}$, where $z = \cos\theta + \mathrm{i}\sin\theta$, show that, if $\sin\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}. \qquad [7]$$

Deduce that

$$\sum_{r=1}^{n}(2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right] = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right). \qquad [4]$$

44. [9231/s15/13/q4]

Use the formula for $\tan(\boldsymbol{A} - \boldsymbol{B})$ in the List of Formulae (MF10) to show that

$$\tan^{-1}(\boldsymbol{x}+1) - \tan^{-1}(\boldsymbol{x}-1) = \tan^{-1}\left(\frac{2}{\boldsymbol{x}^2}\right). \qquad [3]$$

Deduce the sum to $\boldsymbol{n}$ terms of the series

$$\tan^{-1}\left(\frac{2}{1^2}\right) + \tan^{-1}\left(\frac{2}{2^2}\right) + \tan^{-1}\left(\frac{2}{3^2}\right) + \dots. \qquad [4]$$

45. [9231/w15/11/q4]

The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

Chapter 4

Matrices

1. [9231/s25/11/q4]

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $0 < \theta < 2\pi$.

(a) The matrix **M** represents a sequence of two geometrical transformations in the x–y plane.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) Find the value of θ for which the transformation represented by **M** has a line of invariant points. [7]

2. [9231/s25/13/q1]

The matrix **M** represents the sequence of two transformations in the x-y plane given by a stretch parallel to the x-axis, scale factor 14, followed by a rotation anticlockwise about the origin through angle $\frac{1}{3}\pi$.

(a) Show that $2\mathbf{M} = \begin{pmatrix} 14 & -\sqrt{3} \\ 14\sqrt{3} & 1 \end{pmatrix}$. [4]

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **M**. [5]

The unit square S in the x-y plane is transformed by **M** onto the rectangle P.

(c) Find the matrix which transforms P onto S. [2]

3. [9231/s25/14/q5]

The matrix **M** represents a sequence of two transformations in the *x*-*y* plane given by a one-way stretch in the *x*-direction, scale factor 3, followed by a reflection in the line $y = x$.

(a) Find **M**. [3]

(b) Give full details of the geometrical transformation in the *x*-*y* plane represented by $\mathbf{M}^{-1}$. [3]

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

(c) Find **N**. [3]

(d) Find the equations of the invariant lines, through the origin, of the transformation in the *x*-*y* plane represented by **MN**. [5]

4. [9231/w25/11/q2]

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [2]

(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [2]

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

5. [9231/w25/12/q4]

Let k and m be non-zero constants. The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} k & 0 \\ 0 & m \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by the matrix **B** in each of the following cases.

(i) $m = 1$ [2]

(ii) $m = k$ [2]

(b) Show that the matrix **ABC** is singular. [6]

6. [9231/w25/14/q5]

(a) **(i)** By writing $\cos\left(\frac{1}{12}\pi\right)$ as $\cos\left(\frac{1}{4}\pi - \frac{1}{6}\pi\right)$, show that $\cos\left(\frac{1}{12}\pi\right) = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$. [1]

(ii) Show also that $\sin\left(\frac{1}{12}\pi\right) = \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$. [1]

The matrix **M** is such that $\mathbf{M} = \frac{1}{4}\begin{pmatrix} \sqrt{6}+\sqrt{2} & \sqrt{2}-\sqrt{6} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) The matrix **M** represents a sequence of two geometrical transformations in the x-y plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

(c) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is **I**. [2]

(d) Given that $y = mx$ is an invariant line of the transformation represented by **M**, show that

$$m^2 \sin\left(\tfrac{1}{12}\pi\right) + 2m\cos\left(\tfrac{1}{12}\pi\right) - \sin\left(\tfrac{1}{12}\pi\right) = 0$$

and find the values of m in the form $a\cot\left(\frac{1}{12}\pi\right) + b\operatorname{cosec}\left(\frac{1}{12}\pi\right)$, where a and b are integers to be determined. [6]

7. [9231/s24/11/q4]

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 14 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations in the x-y plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

(c) Find the equations of the invariant lines, through the origin, of the transformation represented by $\mathbf{M}$. [5]

(d) The triangle ABC in the x-y plane is transformed by $\mathbf{M}$ onto triangle DEF.

Given that the area of triangle DEF is $28\,\text{cm}^2$, find the area of triangle ABC. [2]

8. [9231/s24/13/q1]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 1 & 0 \\ 6 & 5 & 2 \\ -1 & 3 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

(b) Given that $\mathbf{A}^{-1} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 0 & 0 \\ -\frac{23}{2} & \frac{1}{2} & 3 \end{pmatrix}$, find the value of k. [2]

9. [9231/s24/13/q3]

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations in the x-y plane.

Give full details of each transformation, and make clear the order in which they are applied. [4]

(b) Find the equations of the invariant lines, through the origin, of the transformation represented by $\mathbf{M}$. [5]

The triangle DEF in the x-y plane is transformed by $\mathbf{M}$ onto triangle PQR.

(c) Given that the area of triangle PQR is $35\,\text{cm}^2$, find the area of triangle DEF. [2]

10. [9231/w24/11/q1]

The matrix **M** represents the sequence of two transformations in the x-y plane given by a stretch parallel to the x-axis, scale factor k ($k \neq 0$), followed by a shear, x-axis fixed, with (0, 1) mapped to (k, 1).

(a) Show that $\mathbf{M} = \begin{pmatrix} k & k \\ 0 & 1 \end{pmatrix}$. [4]

(b) The transformation represented by **M** has a line of invariant points.

Find, in terms of k, the equation of this line. [3]

The unit square S in the x-y plane is transformed by **M** onto the parallelogram P.

(c) Find, in terms of k, a matrix which transforms P onto S. [1]

(d) Given that the area of P is $3k^2$ units^2, find the possible values of k. [2]

11. [9231/w24/12/q4]

The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [3]

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

Let $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) Give full details of the transformation represented by **M**. [2]

(d) Find the matrix **N** such that $\mathbf{NM} = \mathbf{CAB}$. [3]

12. [9231/s23/11/q4]

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}$, where a, b, c are real constants and $b \neq 0$.

(a) Show that **M** does not represent a rotation about the origin. [2]

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by **M**. [5]

It is given that **M** represents the sequence of two transformations in the x–y plane given by an enlargement, centre the origin, scale factor 5 followed by a shear, x-axis fixed, with $(0, 1)$ mapped to $(5, 1)$.

(c) Find **M**. [3]

(d) The triangle DEF in the x–y plane is transformed by **M** onto triangle PQR.

Given that the area of triangle DEF is $12\,\text{cm}^2$, find the area of triangle PQR. [2]

13. [9231/s23/13/q4]

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where $0 < \theta < \pi$ and k is a non-zero constant. The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations, one of which is a shear.

(a) Describe fully the other transformation and state the order in which the transformations are applied. [3]

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

(c) Find, in terms of k, the value of $\tan\theta$ for which $\mathbf{M}-\mathbf{I}$ is singular. [5]

(d) Given that $k = 2\sqrt{3}$ and $\theta = \frac{1}{3}\pi$, show that the invariant points of the transformation represented by $\mathbf{M}$ lie on the line $3y+\sqrt{3}x = 0$. [4]

14. [9231/w23/11/q5]

Let k be a constant. The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & k & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

It is given that **A** is singular.

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [5]

(b) Find the equations of the invariant lines, through the origin, of the transformation in the $x-y$ plane represented by **CAB**. [5]

(c) The matrices **D**, **E** and **F** represent geometrical transformations in the $x-y$ plane.

- **D** represents an enlargement, centre the origin.
- **E** represents a stretch parallel to the x-axis.
- **F** represents a reflection in the line $y = x$.

Given that $\mathbf{CAB} = \mathbf{D} - 9\mathbf{EF}$, find **D**, **E** and **F**. [5]

15. [9231/w23/12/q3]

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ and $k \neq 1$.

(a) The matrix **M** represents a sequence of two geometrical transformations. State the type of each transformation, and make clear the order in which they are applied. [2]

The unit square in the x−y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of k, the area of parallelogram $OPQR$ and the matrix which transforms $OPQR$ onto the unit square. [3]

(c) Show that the line through the origin with gradient $\frac{1}{k-1}$ is invariant under the transformation in the x−y plane represented by **M**. [3]

16. [9231/s22/11/q7]

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which **A** is non-singular. [3]

(b) Given that **A** is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix **C** such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

17. [9231/s22/13/q5]

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

18. [9231/w22/11/q5]

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where k is a constant.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) The triangle ABC in the x–y plane is transformed by **M** onto triangle DEF.

Find, in terms of k, the single matrix which transforms triangle DEF onto triangle ABC. [2]

(c) Find the set of values of k for which the transformation represented by **M** has no invariant lines through the origin. [7]

19. [9231/w22/12/q3]

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ or 1.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

(c) Show that the invariant points of the transformation represented by $\mathbf{M}$ lie on the line $y = \dfrac{k^2}{1-k}x$. [4]

(d) The triangle ABC in the x-y plane is transformed by $\mathbf{M}$ onto triangle DEF.

Find the value of k for which the area of triangle DEF is equal to the area of triangle ABC. [2]

20. [9231/s21/11/q4]

The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. [3]

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **MN**. [5]

21. [9231/s21/13/q4]

The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

(b) Given that **A** is singular, find the value of k. [3]

(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

22. [9231/w21/11/q4]

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by $\mathbf{M}$ has exactly one invariant line through the origin, giving your answers in terms of π. [9]

23. [9231/w21/12/q1]

(a) Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

Given that the area of triangle DEF is $13\,\text{cm}^2$, find the area of triangle PQR. [2]

(c) Find the matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by $\mathbf{A}$. [4]

24. [9231/s20/11/q6]

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

25. [9231/s20/13/q4]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

26. [9231/w20/11/q1]

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

The unit square in the x-y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x + 3y = 0$ is invariant under the transformation represented by **M**.

(c) Find the values of a and b. [5]

27. [9231/w20/12/q4]

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [1]

(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [3]

(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

Chapter 5

Polar coordinates

1. [9231/s25/11/q5]

The curve C has polar equation $r = \theta e^{\frac{1}{8}\theta}$, for $0 \leqslant \theta \leqslant 2\pi$.

(a) Sketch C. [2]

(b) Find the area of the region bounded by C and the initial line, giving your answer in the form $\left(p\pi^2 + q\pi + r\right)e^{\frac{1}{2}\pi} + s$, where p, q, r and s are integers to be determined. [6]

(c) Show that, at the point of C furthest from the initial line,

$$\theta\cos\theta + \left(\frac{1}{8}\theta + 1\right)\sin\theta = 0$$

and verify that this equation has a root between 5 and 5.05. [5]

2. [9231/s25/13/q7]

The curve C has polar equation $r^2 = \mathrm{e}^{\sin\theta}\cos\theta$, for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole, giving your answers correct to 3 decimal places. [5]

(b) Find the polar coordinates of the point on C that is furthest from the half-line $\theta = \frac{1}{2}\pi$, giving your answers correct to 3 decimal places. [5]

(c) Sketch C. [3]

(d) Find the area of the region bounded by C, giving your answer in exact form. [3]

3. [9231/s25/14/q6]

The curve C has polar equation $r = a\tan\left(\frac{1}{8}\theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant 2\pi$.

(a) Sketch C and state, in terms of a, the greatest distance of a point on C from the pole. [3]

(b) Find, in terms of a, the area of the region bounded by C and the initial line. [4]

(c) Show that, at the point on C furthest from the initial line,

$$4\sin\left(\frac{1}{4}\theta\right)\cos\theta + \sin\theta = 0$$

and verify that this equation has a root between 4.95 and 5. [6]

4. [9231/w25/11/q6]

The curve C has polar equation $r = \sin 3\theta$, for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(a) Sketch C and state the equation of the line of symmetry. [3]

(b) Find the exact value of the area of the region enclosed by C. [4]

In parts **(c)** and **(d)** you may use the identity $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$.

(c) Find the maximum distance of a point on C from the initial line. [5]

(d) Find a Cartesian equation for C. [3]

5. [9231/w25/12/q5]

The curve C has polar equation $r^2 = \tan 2\theta$, where $0 \leqslant \theta \leqslant \frac{1}{8}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

....

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{8}\pi$. [4]

(c) Show that C has Cartesian equation $x^4 - 2xy - y^4 = 0$ given that $0 \leqslant x \leqslant \cos\left(\frac{1}{8}\pi\right)$ and $0 \leqslant y \leqslant \sin\left(\frac{1}{8}\pi\right)$.

[4]

(d) Using your answer to **(b)**, deduce the exact value of the area bounded by C, the x-axis and the line $x = \cos\left(\frac{1}{8}\pi\right)$. [2]

6. [9231/w25/14/q6]

The curve C has polar equation $r = \cos\frac{1}{2}\theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C. [2]

In parts **(b)** and **(c)** you may use the identities $\sin\theta \equiv 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and $\cos\theta \equiv 2\cos^2\frac{1}{2}\theta - 1$.

(b) Find the exact value of the area of the region enclosed by C and the initial line. [4]

(c) Find the maximum distance of a point on C from the initial line. Give your answer in the form $p\sqrt{q}$, where p and q are rational numbers to be determined. [6]

7. [9231/s24/11/q7]

The curve C has polar equation $r^2 = (\pi - \theta)\tan^{-1}(\pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Using the substitution $u = \pi - \theta$, or otherwise, find the area of the region enclosed by C and the initial line. [7]

(c) Show that, at the point of C furthest from the initial line,

$$2(\pi - \theta)\tan^{-1}(\pi - \theta)\cot\theta - \frac{\pi - \theta}{1 + (\pi - \theta)^2} - \tan^{-1}(\pi - \theta) = 0$$

and verify that this equation has a root for θ between 1.2 and 1.3. [5]

8. [9231/s24/13/q7]

The curve C has polar equation $r^2 = \sin 2\theta \cos\theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the equation of the line of symmetry. [3]

(b) Find a Cartesian equation for C. [3]

(c) Find the total area enclosed by C. [4]

(d) Find the greatest distance of a point on C from the pole. [6]

9. [9231/w24/11/q5]

(a) Show that the curve with Cartesian equation

$$\left(x^2+y^2\right)^2 = 6xy$$

has polar equation $r^2 = 3\sin 2\theta$. [2]

The curve C has polar equation $r^2 = 3\sin 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(b) Sketch C and state the maximum distance of a C from the pole. [3]

(c) Find the area of the region enclosed by C. [2]

(d) Find the maximum distance of a point on C from the initial line. [6]

10. [9231/w24/12/q7]

The curve C_1 has polar equation $r = a(\cos\theta + \sin\theta)$ for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{3}{4}\pi$, where a is a positive constant.

(a) Find a Cartesian equation for C_1 and show that it represents a circle, stating its radius and the Cartesian coordinates of its centre. [4]

(b) Sketch C_1 and state the greatest distance of a point on C_1 from the pole. [3]

The curve C_2 with polar equation $r = a\theta$ intersects C_1 at the pole and the point with polar coordinates $(a\phi, \phi)$.

(c) Verify that $1.25 < \phi < 1.26$. [2]

(d) Show that the area of the smaller region enclosed by C_1 and C_2 is equal to

$$\frac{1}{2}a^2\left(\frac{3}{4}\pi + \frac{1}{3}\phi^3 - \phi + \frac{1}{2}\cos 2\phi\right)$$

and deduce, in terms of a and ϕ, the area of the larger region enclosed by C_1 and C_2. [7]

11. [9231/s23/11/q5]

The curve C has polar equation $r^2 = \dfrac{1}{\theta^2 + 1}$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Find the area of the region enclosed by C, the initial line, and the half-line $\theta = \pi$. [4]

(c) Show that, at the point of C furthest from the initial line,

$$\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

and verify that this equation has a root between 1.1 and 1.2. [5]

12. [9231/s23/13/q5]

(a) Show that the curve with Cartesian equation

$$x^2 - y^2 = a,$$

where a is a positive constant, has polar equation $r^2 = a\sec 2\theta$. [3]

The curve C has polar equation $r^2 = a\sec 2\theta$, where a is a positive constant, for $0 \leqslant \theta < \frac{1}{4}\pi$.

(b) Sketch C and state the minimum distance of C from the pole. [3]

(c) Find, in terms of a, the exact value of the area of the region enclosed by C, the initial line, and the half-line $\theta = \frac{1}{12}\pi$. [You may use any result from the list of formulae (MF19) without proof.] [4]

13. [9231/w23/11/q6]

(a) Show that the curve with Cartesian equation

$$\left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}$$

has polar equation $r=\cos\theta$. [3]

The curves C_1 and C_2 have polar equations

$$r=\cos\theta \quad \text{and} \quad r=\sin 2\theta$$

respectively, where $0\leqslant\theta\leqslant\frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole and at another point P.

(b) Find the polar coordinates of P. [3]

(c) In a single diagram sketch C_1 and C_2, clearly identifying each curve, and mark the point P. [3]

(d) The region R is enclosed by C_1 and C_2 and includes the line OP.

Find, in exact form, the area of R. [6]

14. [9231/w23/12/q6]

The curve C has polar equation $r = \mathrm{e}^{-\theta} - \mathrm{e}^{-\frac{1}{2}\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

(b) Find the exact value of the area of the region bounded by C and the initial line. [5]

(c) Show that, at the point on C furthest from the initial line,

$$1 - \mathrm{e}^{\theta - \frac{1}{2}\pi} - \tan\theta = 0$$

and verify that this equation has a root between 0.56 and 0.57. [5]

15. [9231/s22/11/q6]

The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\frac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

16. [9231/s22/13/q6]

The curve C has Cartesian equation $x^2+xy+y^2=a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2=\dfrac{2a}{2+\sin 2\theta}$. [3]

(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

The region R is enclosed by this part of C, the initial line and the half-line $\theta=\frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\dfrac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\frac{1}{2}a\int_0^{\frac{1}{4}\pi}\frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

and use the substitution $t=\tan\theta$ to find the exact value of this area. [8]

17. [9231/w22/11/q6]

(a) Show that the curve with Cartesian equation

$$\left(x^2+y^2\right)^2 = 36\left(x^2-y^2\right)$$

has polar equation $r^2 = 36\cos 2\theta$. [3]

The curve C has polar equation $r^2 = 36\cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the maximum distance of a point on C from the pole. [3]

(c) Find the area of the region enclosed by C. [2]

(d) Find the maximum distance of a point on C from the initial line, giving the answer in exact form. [6]

18. [9231/w22/12/q5]

The curve C has polar equation $r = a\sec^2\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C, stating the polar coordinates of the point of intersection of C with the initial line and also with the half-line $\theta = \frac{1}{4}\pi$. [3]

(b) Find the maximum distance of a point of C from the initial line. [2]

(c) Find the area of the region enclosed by C, the initial line and the half-line $\theta = \frac{1}{4}\pi$. [4]

(d) Find, in the form $y = \mathrm{f}(x)$, the Cartesian equation of C. [3]

19. [9231/s21/11/q5]

The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

(c) Show that C has Cartesian equation $2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2 + y^2}$. [3]

20. [9231/s21/13/q5]

The curve C has polar equation $r = \frac{1}{\pi - \theta} - \frac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\frac{3 - 4\ln 2}{4\pi}$. [6]

21. [9231/w21/11/q6]

The curve C has polar equation $r = 2\cos\theta(1+\sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

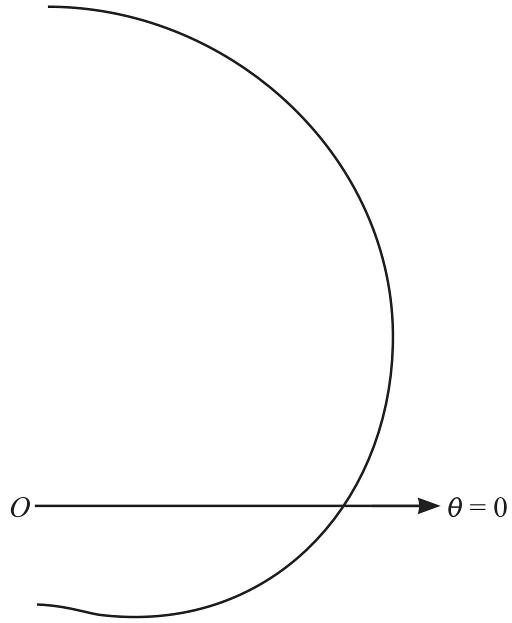
(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

22. [9231/w21/12/q5]

The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

23. [9231/s20/11/q7]

The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

24. [9231/s20/13/q5]

The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$. [2]

25. [9231/w20/11/q7]

(a) Show that the curve with Cartesian equation

$$(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$$

has polar equation $r = \sin 4\theta$. [4]

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

(c) Find the exact value of the area of the region enclosed by C. [4]

(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

26. [9231/w20/12/q5]

The curve C has polar equation $r = \ln(1+\pi-\theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Using the substitution $u = 1+\pi-\theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi)-2\big)+\pi.$$ [6]

(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta)-\tan\theta = 0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

27. [9231/s19/11/q11e]

Answer only **one** of the following two alternatives.

EITHER

The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

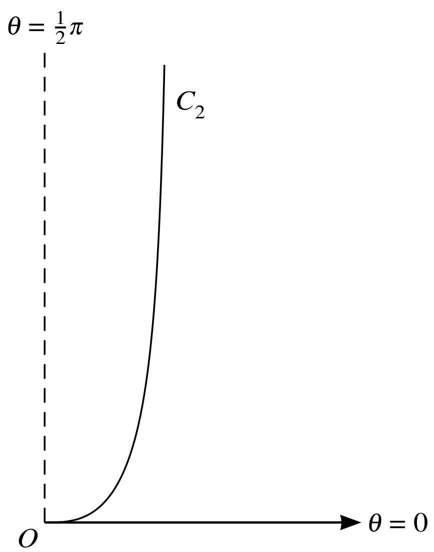
$$2\theta \tan\theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r^2 = \theta \sec^2\theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

28. [9231/s19/13/q2]

The curve C has polar equation $r^2 = \ln(1 + \theta)$, for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

29. [9231/w19/11/q11o]

OR

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2(e^{\theta} + e^{-\theta}),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

30. [9231/s18/11/q3]

The curve C has polar equation $r = \cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Find a cartesian equation of C. [3]

31. [9231/s18/13/q8]

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \pi$, as follows:

$$C_1\colon r = a,$$
$$C_2\colon r = 2a|\cos\theta|,$$

where a is a positive constant. The curves intersect at the points P_1 and P_2.

(i) Find the polar coordinates of P_1 and P_2. [2]

(ii) In a single diagram, sketch C_1, C_2 and their line of symmetry. [3]

(iii) The region R enclosed by C_1 and C_2 is bounded by the arcs OP_1, P_1P_2 and P_2O, where O is the pole. Find the area of R, giving your answer in exact form. [5]

32. [9231/w18/11/q9]

The curve C has polar equation

$$r = 5\sqrt{(\cot\theta)},$$

where $0.01 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Find the area of the finite region bounded by C and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. [3]

Let P be the point on C where $\theta = 0.01$.

(ii) Find the distance of P from the initial line, giving your answer correct to 1 decimal place. [2]

(iii) Find the maximum distance of C from the initial line. [3]

(iv) Sketch C. [2]

33. [9231/w18/12/q3]

The curve C has polar equation $r = a\cos 3\theta$, for $-\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{6}\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Using the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find a cartesian equation of C. [3]

34. [9231/s17/11/q11]

The curve C has polar equation $r = a(1 + \sin\theta)$ for $-\pi < \theta \leqslant \pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C. [4]

(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1 + \sin\theta)}\, \mathrm{d}\theta. \qquad [3]$$

(iv) Show that the substitution $u = 1 + \sin\theta$ reduces this integral for s to $(\sqrt{2})a \int_0^2 \frac{1}{\sqrt{(2-u)}}\, \mathrm{d}u$. Hence evaluate s. [4]

35. [9231/s17/13/q11e]

A curve C has polar equation $r = 2a\cos\left(2\theta + \frac{1}{2}\pi\right)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Show that $r = -2a\sin 2\theta$ and sketch C. [4]

(ii) Deduce that the cartesian equation of C is

$$(x^2 + y^2)^{\frac{3}{2}} = -4axy.$$ [2]

(iii) Find the area of one loop of C. [5]

(iv) Show that, at the points (other than the pole) at which a tangent to C is parallel to the initial line,

$$2\tan\theta = -\tan 2\theta.$$ [3]

36. [9231/w17/11/q11o]

The polar equation of a curve C is $r = a(1 + \cos\theta)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Show that the cartesian equation of C is

$$x^2 + y^2 = a\big(x + \sqrt{(x^2 + y^2)}\big).$$ [2]

(iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

(iv) Find the arc length of C between the point where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$. [5]

37. [9231/s16/11/q4]

A curve C has polar equation $r^2 = 8\,\mathrm{cosec}\,2\theta$ for $0 < \theta < \frac{1}{2}\pi$. Find a cartesian equation of C. [3]

Sketch C. [2]

Determine the exact area of the sector bounded by the arc of C between $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{3}\pi$, the half-line $\theta = \frac{1}{6}\pi$ and the half-line $\theta = \frac{1}{3}\pi$. [3]

[It is given that $\int \mathrm{cosec}\,x\,\mathrm{d}x = \ln\left|\tan\frac{1}{2}x\right| + c$.]

38. [9231/s16/13/q7]

A curve has polar equation $r = \dfrac{1}{1 - \cos\theta}$, for $0 < \theta < 2\pi$. Find, in the form $y^2 = \mathrm{f}(x)$, the cartesian equation of the curve. [3]

Hence sketch the curve, and shade the region whose area is given by $\dfrac{1}{2}\displaystyle\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \dfrac{1}{(1 - \cos\theta)^2}\,\mathrm{d}\theta$. [3]

Using the cartesian equation of the curve, find the area of this region. [3]

39. [9231/s15/11/q5]

The curves C_1 and C_2 have polar equations

$$C_1: \quad r = \frac{1}{\sqrt{2}}, \quad \text{for } 0 \leqslant \theta < 2\pi,$$
$$C_2: \quad r = \sqrt{\left(\sin \tfrac{1}{2}\theta\right)}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

Sketch C_1 and C_2 on the same diagram. [3]

Find the exact value of the area of the region enclosed by C_1, C_2 and the half-line $\theta = 0$. [4]

40. [9231/s15/13/q2]

The curve $\boldsymbol{C}$ has polar equation $\boldsymbol{r} = \mathrm{e}^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of $\boldsymbol{C}$ is 2015. Find the value of α. [6]

41. [9231/w15/11/q11o]

The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = 4a^2\sin^2\left(\tfrac{1}{2}\theta\right),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]

Chapter 6

Vectors

1. [9231/s25/11/q6]

The points A, B, C have position vectors

$$\mathbf{i}-2\mathbf{k}, \quad \mathbf{i}+2\mathbf{j}+2\mathbf{k}, \quad 2\mathbf{i}-\mathbf{j}-\mathbf{k},$$

respectively.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

A point D has position vector $\mathbf{i}+t\mathbf{k}$, where $t \neq -2$.

(b) Find the acute angle between the planes ABC and ABD. [4]

(c) Find the values of t such that the shortest distance between the lines AB and CD is $\sqrt{2}$. [7]

2. [9231/s25/13/q5]

The plane Π has equation $\mathbf{r} = 2\mathbf{i}+3\mathbf{j}-2\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}-\mathbf{k})+\mu(3\mathbf{i}+2\mathbf{j}-2\mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

The point P has position vector $4\mathbf{i}+2\mathbf{j}+9\mathbf{k}$.

(b) Find the position vector of the foot of the perpendicular from P to Π. [4]

(c) The line l is parallel to the vector $3\mathbf{i}+5\mathbf{j}-\mathbf{k}$.

Find the acute angle between l and Π. [3]

3. [9231/s25/14/q3]

The points A, B and C have position vectors

$$2\mathbf{j}+3\mathbf{k}, \quad -5\mathbf{i}+3\mathbf{j}+\mathbf{k} \quad \text{and} \quad \mathbf{i}+2\mathbf{j}+5\mathbf{k}$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the line OA and the plane ABC. [3]

4. [9231/w25/11/q5]

The plane Π_1 has equation $\mathbf{r} = -3\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$. [4]

(b) Find the perpendicular distance from the point with position vector $-\mathbf{i} - 2\mathbf{k}$ to Π_1. [3]

(c) The plane Π_2 has equation $3x + 2y - z = 14$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

5. [9231/w25/12/q6]

The plane Π has equation $x+3y+2z=1$.

(a) Find the perpendicular distance from the origin O to the plane Π. [2]

Relative to O, the points A, B, C have position vectors

$$-\mathbf{j}+2\mathbf{k},\quad 2\mathbf{i}-\mathbf{k},\quad 2\mathbf{i}-\mathbf{j}-\mathbf{k},$$

respectively.

(b) Find the acute angle between the planes OAB and Π. [4]

(c) Find an equation for the common perpendicular to the lines OC and AB. [8]

6. [9231/w25/14/q4]

The points A, B, C have position vectors

$$3\mathbf{i}+5\mathbf{j}+5\mathbf{k}, \qquad 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \qquad 2\mathbf{i}+\mathbf{j}-\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) The point D has position vector $\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

Find the shortest distance between the lines AB and CD. [5]

7. [9231/s24/11/q5]

The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \qquad 2\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad -3\mathbf{i}-3\mathbf{j}+4\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

The point D has position vector $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

(b) Find the perpendicular distance from D to the plane ABC. [2]

(c) Find the shortest distance between the lines AB and CD. [5]

8. [9231/s24/13/q5]

The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = -3\mathbf{i} + 4\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Obtain an equation of Π_1 in the form $px + qy + rz = s$. [2]

(c) The point $(1, 1, 1)$ lies on the plane Π_2.

It is given that the line of intersection of the planes Π_1 and Π_2 passes through the point $(0, 0, 2)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$.

Obtain an equation of Π_2 in the form $ax + by + cz = d$. [3]

9. [9231/w24/11/q7]

The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 9\mathbf{k} + \mu(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ respectively. The plane Π_1 contains l_1 and is parallel to l_2.

(a) Find the equation of Π_1, giving your answer in the form $ax + by + cz = d$. [4]

The plane Π_2 contains l_2 and the point with coordinates $(2, -1, 7)$.

(b) Find the acute angle between Π_1 and Π_2. [4]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Find a vector equation for PQ. [7]

10. [9231/w24/12/q2]

The line l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 4\mathbf{k})$.

The plane Π contains l_1 and is parallel to the vector $2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l_2 is parallel to the vector $5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$.

(b) Find the acute angle between l_2 and Π. [3]

11. [9231/s23/11/q7]

The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

The line l passes through the point A with position vector $a\mathbf{i} + a\mathbf{j} + (a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i} + b\mathbf{j} + b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

12. [9231/s23/13/q6]

The points A, B, C have position vectors

$$\mathbf{i}+\mathbf{j}, \quad -\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find a vector equation of the common perpendicular to the lines OC and AB. [8]

13. [9231/w23/11/q4]

The lines l_1 and l_2 have equations

$$\mathbf{r} = -2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} + \lambda(-4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and the point with position vector $-\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$.

(b) Find an equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

14. [9231/w23/12/q5]

The plane Π_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$. [4]

The line l, which does not lie in Π_1, has equation $\mathbf{r} = -3\mathbf{i} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

(b) Show that l is parallel to Π_1. [2]

(c) Find the distance between l and Π_1. [3]

(d) The plane Π_2 has equation $3x + 3y + 2z = 1$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

15. [9231/s22/11/q2]

The points A, B, C have position vectors

$$4\mathbf{i}-4\mathbf{j}+\mathbf{k}, \quad -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}, \quad 4\mathbf{i}-\mathbf{j}-2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

16. [9231/s22/13/q7]

The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

Let Π_1 be the plane ABD when $\lambda=1$.

Let Π_2 be the plane ABD when $\lambda=4$.

(b) (i) Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+s\mathbf{b}+t\mathbf{c}$. [2]

(ii) Find an equation of Π_2, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

17. [9231/w22/11/q4]

The plane Π contains the lines $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and is parallel to the vector $\mathbf{j} + \mathbf{k}$

(b) Find the acute angle between l and Π. [3]

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

18. [9231/w22/12/q6]

The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) (i) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

19. [9231/s21/11/q6]

Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

The plane Π_2 has Cartesian equation $5x-6y+7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]

(d) Find the acute angle between Π_1 and Π_2. [3]

20. [9231/s21/13/q6]

The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

21. [9231/w21/11/q5]

The plane Π has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

22. [9231/w21/12/q7]

The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r.n} = p$. [3]

The plane Π has equation $x-3y-2z = 1$.

(b) Find the perpendicular distance of Π from the origin. [1]

(c) Find the acute angle between the planes OAB and Π. [3]

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

23. [9231/s20/11/q5]

The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and is parallel to the vector $\mathbf{i} + \mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

24. [9231/s20/13/q7]

The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(b) Find the distance between l_2 and Π. [3]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

25. [9231/w20/11/q4]

The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

26. [9231/w20/12/q7]

The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the acute angle between the planes OBC and ABC. [4]

The point D has position vector $t\mathbf{i}-\mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

27. [9231/s19/11/q3]

The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

28. [9231/s19/13/q7]

The line l_1 passes through the points $A\,(-3,\ 1,\ 4)$ and $B\,(-1,\ 5,\ 9)$. The line l_2 passes through the points $C\,(-2,\ 6,\ 5)$ and $D\,(-1,\ 7,\ 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

29. [9231/w19/11/q6]

With O as the origin, the points A, B, C have position vectors

$$\mathbf{i} - \mathbf{j}, \quad 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}, \quad \mathbf{i} - \mathbf{j} + \mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

30. [9231/s18/11/q10]

The line l_1 is parallel to the vector $a\mathbf{i} - \mathbf{j} + \mathbf{k}$, where a is a constant, and passes through the point whose position vector is $9\mathbf{j} + 2\mathbf{k}$. The line l_2 is parallel to the vector $-a\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and passes through the point whose position vector is $-6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a = -\frac{6}{13}$. [3]

(b) Find a cartesian equation of the plane containing l_1 and l_2. [4]

(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{(30)}$, find the value of a. [5]

31. [9231/s18/13/q7]

The lines l_1 and l_2 have vector equations

$$\mathbf{r} = a\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a. [3]

The point P has position vector $3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. [4]

(iii) Find the perpendicular distance from P to l_2. [4]

32. [9231/w18/11/q8]

The plane Π_1 has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $3x + y - z = 3$.

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [2]

(iii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

33. [9231/w18/12/q10]

The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \qquad 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}, \qquad -\mathbf{i} + 3\mathbf{k}, \qquad m\mathbf{j} + 4\mathbf{k},$$

respectively, where m is a constant.

(i) Show that the lines AB and CD are parallel when $m = \frac{3}{2}$. [1]

(ii) Given that $m \neq \frac{3}{2}$, find the shortest distance between the lines AB and CD. [5]

(iii) When $m = 2$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [6]

34. [9231/s17/11/q12o]

The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad 5\mathbf{i} - 5\mathbf{j} + \alpha\mathbf{k},$$

respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$.

(i) Show that the possible values of α are 3 and 5. [7]

(ii) Using $\alpha = 3$, find the shortest distance of the point D from the line AC, giving your answer correct to 3 significant figures. [3]

(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [4]

35. [9231/s17/13/q9]

The plane Π_1 passes through the points $(1, 2, 1)$ and $(5, -2, 9)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(i) Find the cartesian equation of Π_1. [4]

The plane Π_2 contains the lines

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).$$

(ii) Find the cartesian equation of Π_2. [4]

(iii) Find the acute angle between Π_1 and Π_2. [3]

36. [9231/w17/11/q6]

The points A, B and C have position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ respectively.

(i) Find the area of the triangle ABC. [4]

(ii) Find the perpendicular distance of the point A from the line BC. [3]

(iii) Find the cartesian equation of the plane through A, B and C. [2]

37. [9231/s16/11/q8]

Find a cartesian equation of the plane Π_1 passing through the points with coordinates $(2, -1, 3)$, $(4, 2, -5)$ and $(-1, 3, -2)$. [4]

The plane Π_2 has cartesian equation $3x - y + 2z = 5$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of the planes Π_1 and Π_2. [4]

38. [9231/s16/13/q11o]

The position vectors of the points A, B, C, D are

$$\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{d} = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$$

respectively. It is given that the shortest distance between the lines AB and CD is 3.

(i) Show that $\lambda^2 + \lambda - 20 = 0$. [7]

(ii) The planes p_1 and p_2 are the planes through A, B and D corresponding to the two values of λ satisfying the equation in part **(i)**. Find the acute angle between p_1 and p_2. [7]

39. [9231/w16/11/q11e]

The lines l_1 and l_2 have equations

$$\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + s(3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Show that the position vector of P is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and find the position vector of Q. [7]

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane Π which passes through P and is perpendicular to l_1. [3]

The plane Π meets the plane $\mathbf{r} = p\mathbf{i} + q\mathbf{j}$ in the line l_3. Find a vector equation of l_3. [4]

40. [9231/s15/11/q11o]

The lines l_1 and l_2 have equations $\mathbf{r} = 8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j})$ and $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k} + \mu(2\mathbf{j} - 3\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vector of the point P and the position vector of the point Q. [8]

The points with position vectors $8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k}$ are denoted by A and B respectively. Find

(i) $\overrightarrow{AP} \times \overrightarrow{AQ}$ and hence the area of the triangle APQ,

(ii) the volume of the tetrahedron $APQB$. (You are given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.)

[6]

41. [9231/s15/13/q8]

A line, passing through the point $\boldsymbol{A}(3, 0, 2)$, has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. It meets the plane Π, which has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3$, at the point $\boldsymbol{P}$. Find the coordinates of $\boldsymbol{P}$. [3]

Write down a vector $\mathbf{n}$ which is perpendicular to Π, and calculate the vector $\mathbf{w}$, where

$$\mathbf{w} = \mathbf{n} \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$ [3]

The point $\boldsymbol{Q}$ lies in Π and is the foot of the perpendicular from $\boldsymbol{A}$ to Π. Use the vector $\mathbf{w}$ to determine an equation of the line $\boldsymbol{PQ}$ in the form $\mathbf{r} = \mathbf{u} + \mu\mathbf{v}$. [4]

42. [9231/w15/11/q11e]

The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q. Find a vector perpendicular to the plane ABC and show that the length of ON is $\dfrac{4}{\sqrt{(21)}}$. [4]

Find the position vector of the point Q. [5]

Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}\left(\frac{2}{3}\right)$. [5]

Chapter 7

Proof by induction

1. [9231/s25/11/q3]

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = 6u_n + 5$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 6^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

2. [9231/s25/13/q2]

Prove by mathematical induction that $2025^n + 47^n - 2$ is divisible by 46 for all positive integers n. [6]

3. [9231/s25/14/q2]

Prove by mathematical induction that, for every integer $n \geqslant 2$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x \ln x) = (-1)^n (n-2)!\,x^{1-n}. \qquad [6]$$

4. [9231/w25/11/q3]

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}(x\cos x) = (-1)^n(x\sin x - (2n-1)\cos x). \qquad [7]$$

5. [9231/w25/12/q3]

The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 5$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{6u_n + 5}{u_n + 2}.$$

(a) By considering $5 - u_{n+1}$, prove by mathematical induction that $u_n < 5$ for all positive integers n. [5]

(b) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

6. [9231/w25/14/q1]

Prove by mathematical induction that $\left(\frac{6}{5}\right)^n \geqslant 1 + \frac{1}{5}n$ for all positive integers n. [5]

7. [9231/s24/11/q2]

Prove by mathematical induction that $6^{4n}+38^{n}-2$ is divisible by 74 for all positive integers n. [6]

8. [9231/w24/11/q2]

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\tan^{-1}x\right) = P_n(x)\left(1+x^2\right)^{-n},$$

where $P_n(x)$ is a polynomial of degree $n-1$. [6]

9. [9231/w24/12/q1]

The sequence u_1, u_2, u_3, … is such that $u_1 = 4$ and $u_{n+1} = 3u_n - 2$ for $n \geqslant 1$.

Prove by induction that $u_n = 3^n + 1$ for all positive integers n. [5]

10. [9231/s23/11/q1]

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$2\mathbf{A}^n = \begin{pmatrix} 2 \times 3^n & 0 \\ 3^n - 1 & 2 \end{pmatrix}.$$ [5]

(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

11. [9231/s23/13/q1]

Prove by mathematical induction that, for all positive integers n, $5^{3n} + 32^n - 33$ is divisible by 31. [6]

12. [9231/w23/11/q2]

Prove by mathematical induction that, for all positive integers n,

$$1+2x+3x^2+\ldots+nx^{n-1}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}. \qquad [6]$$

13. [9231/w23/12/q2]

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right) = \left(x^2 + 2nx + n(n-1)\right)\mathrm{e}^x.$$ [6]

14. [9231/s22/11/q3]

The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

15. [9231/s22/13/q5.b]

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by **A**. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

16. [9231/w22/11/q2]

Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^n - 50$ is divisible by 48. [6]

17. [9231/w22/12/q4]

The function f is such that $f''(x) = f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}\big(x\,f(x)\big) = x\,f'(x) + (2n-1)\,f(x). \qquad [7]$$

18. [9231/s21/11/q1]

Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers n. [6]

19. [9231/s21/13/q3]

(a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4+r^2\right)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

20. [9231/w21/11/q3]

The sequence of real numbers a_1, a_2, a_3, … is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

21. [9231/w21/12/q2]

It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right)e^{ax}.$$ [6]

22. [9231/s20/11/q6.b]

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

23. [9231/s20/13/q2]

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

24. [9231/w20/11/q5]

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\big(x\cos x + (2n-1)\sin x\big). \qquad [7]$$

25. [9231/w20/12/q2]

Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

26. [9231/s19/13/q1]

Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

27. [9231/w19/11/q2]

It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{d^n y}{dx^n} = (-1)^{n-1} \frac{(n-1)!a^n}{(ax+1)^n}.$$ [6]

28. [9231/s18/11/q2]

It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

29. [9231/s17/11/q2]

Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

30. [9231/s17/13/q3]

Prove, by mathematical induction, that $\displaystyle\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

31. [9231/w17/11/q3]

(i) Show that $\dfrac{d^{n+1}}{dx^{n+1}}(x^{n+1}\ln x) = \dfrac{d^n}{dx^n}\left(x^n + (n+1)x^n \ln x\right)$. [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \dots + \frac{1}{n}\right).$$ [5]

32. [9231/s16/11/q3]

Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

33. [9231/s16/13/q2]

It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

34. [9231/w16/11/q4]

Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

35. [9231/s15/11/q3]

The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n. Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

36. [9231/s15/13/q3]

Prove by mathematical induction that, for all positive integers $\boldsymbol{n}$, $\displaystyle\sum_{\boldsymbol{r}=1}^{\boldsymbol{n}} \frac{1}{(2\boldsymbol{r})^2 - 1} = \frac{\boldsymbol{n}}{2\boldsymbol{n}+1}$. [6]

State the value of $\displaystyle\sum_{\boldsymbol{r}=1}^{\infty} \frac{1}{(2\boldsymbol{r})^2 - 1}$. [1]

37. [9231/w15/11/q3]

Given that a is a constant, prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{ax}) = na^{n-1}\mathrm{e}^{ax} + a^n x\mathrm{e}^{ax}.$$ [6]

Formula Sheet MF19

Cambridge Assessment
International Education

List MF19

List of formulae and statistical tables

Cambridge International AS & A Level
Mathematics (9709) and Further Mathematics (9231)

For use from 2020 in all papers for the above syllabuses.

CST319

2508709701

Edited by Thoridal

PURE MATHEMATICS

Mensuration

Volume of sphere = $\frac{4}{3}\pi r^3$

Surface area of sphere = $4\pi r^2$

Volume of cone or pyramid = $\frac{1}{3} \times$ base area $\times$ height

Area of curved surface of cone = $\pi r \times$ slant height

Arc length of circle $= r\theta$ (θ in radians)

Area of sector of circle $= \frac{1}{2} r^2 \theta$ (θ in radians)

Algebra

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d \,, \qquad S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}, \qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1), \qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \ldots + b^n \text{, where } n \text{ is a positive integer}$$

$$\text{and } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{, where } n \text{ is rational and } |x| < 1$$

Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1} x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
uv	$v\frac{du}{dx} + u\frac{dv}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(\boldsymbol{x})$	$\int \mathbf{f}(\boldsymbol{x})\,\mathbf{d}\boldsymbol{x}$	
x^n	$\dfrac{x^{n+1}}{n+1}$	$(n \neq -1)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$	$(\lvert x\rvert < a)$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x = \ln\lvert \mathrm{f}(x)\rvert$$

Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

FURTHER PURE MATHEMATICS

Algebra

Summations:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Trigonometry

If $t = \tan\tfrac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x \equiv 1, \qquad \sinh 2x \equiv 2\sinh x \cosh x, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2\, x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$	
$\sec x$	$\ln\lvert \sec x + \tan x\rvert = \ln\lvert \tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi)\rvert$	$(\lvert x\rvert < \tfrac{1}{2}\pi)$
$\text{cosec}\, x$	$-\ln\lvert \text{cosec}\, x + \cot x\rvert = \ln\lvert \tan(\tfrac{1}{2}x)\rvert$	$(0 < x < \pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\text{sech}^2\, x$	$\tanh x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	$(x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	

MECHANICS

Uniformly accelerated motion

$$v = u + at\,, \qquad s = \tfrac{1}{2}(u+v)t\,, \qquad s = ut + \tfrac{1}{2}at^2\,, \qquad v^2 = u^2 + 2as$$

FURTHER MECHANICS

Motion of a projectile

Equation of trajectory is:

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

Elastic strings and springs

$$T = \frac{\lambda x}{l}\,, \qquad E = \frac{\lambda x^2}{2l}$$

Motion in a circle

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$$\omega^2 r \quad \text{or} \quad \frac{v^2}{r}$$

Centres of mass of uniform bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\frac{3}{8}r$ from centre

Hemispherical shell of radius r: $\frac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\dfrac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\dfrac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\frac{3}{4}h$ from vertex

PROBABILITY & STATISTICS

Summary statistics

For ungrouped data:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

For grouped data:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

Discrete random variables

$$\mathrm{E}(X) = \Sigma xp, \qquad \mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2$$

For the binomial distribution $\mathrm{B}(n, p)$:

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \qquad \mu = np, \qquad \sigma^2 = np(1-p)$$

For the geometric distribution Geo(p):

$$p_r = p(1-p)^{r-1}, \qquad \mu = \frac{1}{p}$$

For the Poisson distribution $\mathrm{Po}(\lambda)$

$$p_r = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!}, \qquad \mu = \lambda, \qquad \sigma^2 = \lambda$$

Continuous random variables

$$\mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{Var}(X) = \int x^2\,\mathrm{f}(x)\,\mathrm{d}x - \{\mathrm{E}(X)\}^2$$

Sampling and testing

Unbiased estimators:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1} = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Central Limit Theorem:

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$\mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$$

FURTHER PROBABILITY & STATISTICS

Sampling and testing

Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \overline{x}_1)^2 + \Sigma(x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$$

Probability generating functions

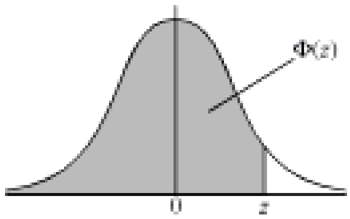
$\mathrm{G}_X(t) = \mathrm{E}(t^X)$, $\mathrm{E}(X) = \mathrm{G}'_X(1)$, $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \{\mathrm{G}'_X(1)\}^2$

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1, then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = \mathrm{P}(Z \leqslant z).$$

For negative values of z, use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1, then, for each value of p, the table gives the value of z such that

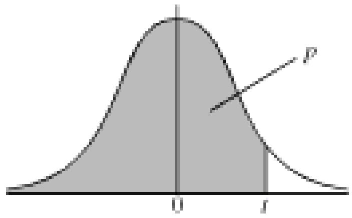
$$\mathrm{P}(Z \leqslant z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE *t*-DISTRIBUTION

If T has a t-distribution with ν degrees of freedom, then, for each pair of values of p and ν, the table gives the value of t such that:

$P(T \leqslant t) = p$.

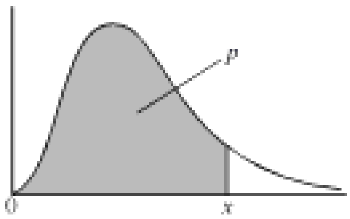


p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
$\nu = 1$	1.000	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.894	6.869
6	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.689
28	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.660
30	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE χ^2-DISTRIBUTION

If X has a χ^2-distribution with ν degrees of freedom then, for each pair of values of p and ν, the table gives the value of x such that

$$P(X \leqslant x) = p.$$



p	0.01	0.025	0.05	0.9	0.95	0.975	0.99	0.995	0.999
$\nu = 1$	$0.0^3 1571$	$0.0^3 9821$	$0.0^2 3932$	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

WILCOXON SIGNED-RANK TEST

The sample has size n.

P is the sum of the ranks corresponding to the positive differences.

Q is the sum of the ranks corresponding to the negative differences.

T is the smaller of P and Q.

For each value of n the table gives the **largest** value of T which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *T*

	Level of significance			
One-tailed Two-tailed	0.05 0.1	0.025 0.05	0.01 0.02	0.005 0.01
$n = 6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

For larger values of n, each of P and Q can be approximated by the normal distribution with mean $\frac{1}{4}n(n+1)$ and variance $\frac{1}{24}n(n+1)(2n+1)$.

WILCOXON RANK-SUM TEST

The two samples have sizes m and n, where $m \leqslant n$.

R_m is the sum of the ranks of the items in the sample of size m.

W is the smaller of R_m and $m(n + m + 1) - R_m$.

For each pair of values of m and n, the table gives the **largest** value of W which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of W

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 3$			$m = 4$			$m = 5$			$m = 6$		
3	6	–	–									
4	6	–	–	11	10	–						
5	7	6	–	12	11	10	19	17	16			
6	8	7	–	13	12	11	20	18	17	28	26	24
7	8	7	6	14	13	11	21	20	18	29	27	25
8	9	8	6	15	14	12	23	21	19	31	29	27
9	10	8	7	16	14	13	24	22	20	33	31	28
10	10	9	7	17	15	13	26	23	21	35	32	29

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 7$			$m = 8$			$m = 9$			$m = 10$		
7	39	36	34									
8	41	38	35	51	49	45						
9	43	40	37	54	51	47	66	62	59			
10	45	42	39	56	53	49	69	65	61	82	78	74

For larger values of m and n, the normal distribution with mean $\frac{1}{2}m(m+n+1)$ and variance $\frac{1}{12}mn(m+n+1)$ should be used as an approximation to the distribution of R_m.

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Syllabus 26-27 Further Pure Mathematics 1

1 Further Pure Mathematics 1 (for Paper 1)

1.1 Roots of polynomial equations

Candidates should be able to:	Notes and examples
• recall and use the relations between the roots and coefficients of polynomial equations	e.g. to evaluate symmetric functions of the roots or to solve problems involving unknown coefficients in equations; restricted to equations of degree 2, 3 or 4 only.
• use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation	Substitutions will not be given for the easiest cases, e.g. where the new roots are reciprocals or squares or a simple linear function of the old roots.

1.2 Rational functions and graphs

Candidates should be able to:	Notes and examples
• sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the demoninator are at most 2	Including determination of the set of values taken by the function, e.g. by the use of a discriminant. Detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as turning points, asymptotes and intersections with the axes.
• understand and use relationships between the graphs of $y = \mathrm{f}(x)$, $y2 = \mathrm{f}(x)$, $y = \frac{1}{\mathrm{f}(x)}$, $y = \|\mathrm{f}(x)\|$ and $y = \mathrm{f}(\|x\|)$.	Including use of such sketch graphs in the course of solving equations or inequalities.

1.3 Summation of series

Candidates should be able to:	Notes and examples
• use the standard results for Σr, Σr^2, Σr^3 to find related sums	
• use the method of differences to obtain the sum of a finite series	Use of partial fractions to express a general term in a suitable form may be required.
• recognise, by direct consideration of a sum to n terms, when a series is convergent, and find the sum to infinity in such cases	

1 Further Pure Mathematics 1

1.4 Matrices

Candidates should be able to:	Notes and examples
• carry out operations of matrix addition, subtraction and multiplication, and recognise the terms zero matrix and identity (or unit) matrix	Including non-square matrices. Matrices will have at most 3 rows and columns.
• recall the meaning of the terms 'singular' and 'non-singular' as applied to square matrices and, for 2×2 and 3×3 matrices, evaluate determinants and find inverses of non-singular matrices	The notations $\det \mathbf{M}$ for the determinant of a matrix $\mathbf{M}$, and $\mathbf{I}$ for the identity matrix, will be used.
• understand and use the result, for non-singular matrices, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$	Extension to the product of more than two matrices may be required.
• understand the use of 2×2 matrices to represent certain geometric transformations in the x-y plane, in particular – understand the relationship between the transformations represented by $\mathbf{A}$ and $\mathbf{A}^{-1}$ – recognise that the matrix product $\mathbf{AB}$ represents the transformation that results from the transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$ – recall how the area scale factor of a transformation is related to the determinant of the corresponding matrix – find the matrix that represents a given transformation or sequence of transformations	Understanding of the terms 'rotation', 'reflection', 'enlargement', 'stretch' and 'shear' for 2D transformations will be required. Other 2D transformations may be included, but no particular knowledge of them is expected.
• understand the meaning of 'invariant' as applied to points and lines in the context of transformations represented by matrices, and solve simple problems involving invariant points and invariant lines.	e.g. to locate the invariant points of the transformation represented by $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$, or to find the invariant lines through the origin for $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, or to show that any line with gradient 1 is invariant for $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

CHAPTER 7. PROOF BY INDUCTION

1 **Further Pure Mathematics 1**

1.5 Polar coordinates

Candidates should be able to:	Notes and examples
• understand the relations between Cartesian and polar coordinates, and convert equations of curves from Cartesian to polar form and vice versa	The convention $r \geqslant 0$ will be used.
• sketch simple polar curves, for $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$ or a subset of either of these intervals	Detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as symmetry, coordinates of intersections with the initial line, the form of the curve at the pole and least/greatest values of r.
• recall the formula $\frac{1}{2}\int r^2 \mathrm{d}\theta$ for the area of a sector, and use this formula in simple cases.	

1.6 Vectors

Candidates should be able to:	Notes and examples
• use the equation of a plane in any of the forms $ax + by + cz = d$ or $\mathbf{r.n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ and convert equations of planes from one form to another as necessary in solving problems	
• recall that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors can be expressed either as $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector, or in component form as $(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$	
• use equations of lines and planes, together with scalar and vector products where appropriate, to solve problems concerning distances, angles and intersections, including – determining whether a line lies in a plane, is parallel to a plane or intersects a plane, and finding the point of intersection of a line and a plane when it exists – finding the foot of the perpendicular from a point to a plane – finding the angle between a line and a plane, and the angle between two planes – finding an equation for the line of intersection of two planes – calculating the shortest distance between two skew lines – finding an equation for the common perpendicular to two skew lines.	

1 Further Pure Mathematics 1

1.7 Proof by induction

Candidates should be able to:	Notes and examples
• use the method of mathematical induction to establish a given result	e.g. $\sum_{r=1}^{n} r^4 = \frac{1}{4}n^2(n+1)^2$, $u_n = \frac{1}{2}(1 + 3^{n-1})$ for the sequence given by $u_{n+1} = 3u_n - 1$ and $u_1 = 1$, $\begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}^n = \begin{pmatrix} 3 \times 2^n - 2 & 1 - 2^n \\ 3 \times 2^{n+1} - 6 & 3 - 2^{n+1} \end{pmatrix}$, $3^{2n} + 2 \times 5^n - 3$ is divisible by 8.
• recognise situations where conjecture based on a limited trial followed by inductive proof is a useful strategy, and carry this out in simple cases.	e.g. find the nth derivative of $x\mathrm{e}^x$, find $\sum_{r=1}^{n} r \times r!$.