The Question Bank of

Further Mechanics

for CAIE 9231 paper 3.

v1.0

Edited by Thoridal

Instructions for Use

- This question bank is organized by chapter for systematic revision.
- This question bank is complied based on the 26-27 CAIE Further Mechanics syllabus, which is included as appendix.
- Each question includes its source for reference.
- Mark schemes are provided in the separate answer booklet.
- The formula sheet (MF19) is included as appendix.
- Use this resource for targeted practice and exam preparation.

Edited by Y.Liu

Contents

Chapter 1

Motion of a projectile

1. [9231/s25/31/q2]

A particle P is projected with speed $24\,\mathrm{m\,s^{-1}}$ at an angle $\theta°$ above the horizontal from a point O on a horizontal plane, and moves freely under gravity. At a horizontal distance 35 m from O, there is a vertical wall of height 10 m which is perpendicular to the vertical plane of motion of P.

(a) Determine the two values of θ for which P just clears the wall. [4]

(b) Given that P clears the wall, find the minimum distance from point O where P can land. [2]

2. [9231/s25/33/q7]

A particle P is projected from a point O with speed U at an angle 45° above the horizontal and moves freely under gravity.

(a) State the vertical and horizontal components of velocity at time t. [1]

At time T, particle P is moving at an angle of 60° below the horizontal.

(b) Show that $T = \frac{U}{2g}(\sqrt{2} + \sqrt{6})$. [3]

At time T, the particle strikes a smooth horizontal plane at a point which is a horizontal distance D from O and a vertical distance H below O.

(c) Find the ratio $H : D$. [4]

After striking the horizontal plane, P rebounds with speed w. The coefficient of restitution between P and the plane is $\frac{2}{3}$.

(d) Find w in terms of U. [3]

3. [9231/s25/34/q3]

A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at any subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha.$$ [4]

It is given that $u = 20\sqrt{2}\,\mathrm{m\,s^{-1}}$ and that the particle P passes through the point where $x = 64\,\mathrm{m}$ and $y = 8\,\mathrm{m}$.

(b) Find the possible values of $\tan\alpha$. [3]

4. **[9231/w25/31/q7]**

A particle P is projected from a point O on a horizontal plane and moves freely under gravity. The initial velocity of P is $25\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. At point A, the direction of motion of P makes an angle of $45°$ with the downward vertical through A.

(a) By differentiating the equation of the trajectory or otherwise, find the coordinates of A. [5]

At point A, the particle strikes a fixed smooth barrier, rebounds, and lands on the horizontal plane. The barrier is inclined at an angle of $45°$ to the horizontal.

(b) Find the speed of P immediately before it collides with the barrier. [3]

(c) Given that the coefficient of restitution between the barrier and the particle is $\frac{1}{9}$, find the horizontal distance travelled by P after it strikes the barrier. [4]

5. [9231/w25/32/q4]

A particle Q is initially positioned at a distance d vertically above a particle P. Particle P is projected with speed U at an angle α above the horizontal. At the same time, Q is projected at an angle β below the horizontal. Both particles move freely under gravity. The particles collide at time T after the projections.

(a) Show that $d = UT(\sin\alpha + \cos\alpha\tan\beta)$. [4]

....

The particles collide when P is at its maximum height.

(b) Given that $\alpha = 30°$ and $\beta = 60°$, find d in terms of U and g. [3]

6. [9231/w25/34/q2]

A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ, where $\tan\theta = 2$, above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a time t s are denoted by x m and y m respectively.

(a) Use the equation of the trajectory given in the list of formulae (MF 19) to show that

$$y = 2x - \frac{25x^2}{u^2}.$$

[1]

(b) In the subsequent motion, P passes through the point with coordinates (8,12). The particle then hits a fixed vertical barrier 7 m high that is at a horizontal distance of D m from the point of projection.

Find the set of possible values of D. [5]

7. [9231/s24/31/q3]

At time $t = 0$ seconds, a particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle $60°$ above the horizontal from a point O. In the subsequent motion P moves freely under gravity. The direction of motion of P when $t = 5$ is perpendicular to its direction of motion when $t = 15$.

Find the value of u. [5]

8. [9231/s24/33/q6]

A particle P is projected with speed $u\ \mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O and moves freely under gravity. After 5 seconds the speed of P is $\frac{3}{4}u$.

(a) Show that $\frac{7}{16}u^2 - 100u\sin\theta + 2500 = 0$. [3]

(b) It is given that the velocity of P after 5 seconds is perpendicular to the initial velocity.

Find, in either order, the value of u and the value of $\sin\theta$. [5]

9. [9231/w24/31/q1]

A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle $\tan^{-1}2$ above the horizontal from a point O on a horizontal plane and moves freely under gravity. When P has travelled a distance 56 m horizontally from O, it is at a vertical height H m above the plane. When P has travelled a distance 84 m horizontally from O, it is at a vertical height $\frac{1}{2}H$ m above the plane.

Find, in either order, the value of u and the value of H. [5]

10. [9231/w24/32/q5]

A particle P is projected from a point O on horizontal ground with speed u at an angle θ above the horizontal, where $\tan\theta = \frac{1}{3}$. The particle P moves freely under gravity and passes through the point with coordinates $(3a, \frac{4}{5}a)$ relative to horizontal and vertical axes through O in the plane of the motion.

(a) Use the equation of the trajectory to show that $u^2 = 25ag$. [2]

At the instant when P is moving horizontally, a particle Q is projected from O with speed V at an angle α above the horizontal. The particles P and Q reach the ground at the same point and at the same time.

(b) Express V^2 in the form kag, where k is a rational number. [6]

11. [9231/s23/31/q7]

At time t s, a particle P is projected with speed $40\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The greatest height achieved by P during its flight is H m and the corresponding time is T s.

(a) Obtain expressions for H and T in terms of θ. [2]

During the time between $t = T$ and $t = 3$, P descends a distance $\frac{1}{4}H$.

(b) Find the value of θ. [4]

(c) Find the speed of P when $t = 3$. [3]

12. [9231/s23/33/q7]

The points O and P are on a horizontal plane, a distance 8 m apart. A ball is thrown from O with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. At the same instant, a model aircraft is launched with speed $5\,\mathrm{m\,s^{-1}}$ parallel to the horizontal plane from a point 4 m vertically above P. The model aircraft moves in the same vertical plane as the ball and in the same horizontal direction as the ball. The model aircraft moves horizontally with a constant speed of $5\,\mathrm{m\,s^{-1}}$. After T s, the ball and the model aircraft collide.

(a) Find the value of T. [6]

(b) Find the direction in which the ball is moving immediately before the collision. [3]

13. [9231/w23/31/q5]

A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. During its flight P passes through the point which is a horizontal distance $3a$ from O and a vertical distance $\frac{3}{8}a$ above the horizontal plane. It is given that $\tan\theta = \frac{1}{3}$.

(a) Show that $u^2 = 8ag$. [2]

A particle Q is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal from O at the instant when P is at its highest point. Particles P and Q both land at the same point on the horizontal plane at the same time.

(b) Find V in terms of a and g. [7]

14. [9231/w23/32/q6]

A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha.$$ [3]

During its flight, P must clear an obstacle of height h m that is at a horizontal distance of 32 m from the point of projection. When $u = 40\sqrt{2}\,\text{m s}^{-1}$, P just clears the obstacle. When $u = 40\,\text{m s}^{-1}$, P only achieves 80% of the height required to clear the obstacle.

(b) Find the two possible values of h. [6]

15. [9231/s22/31/q7]

Particles P and Q are projected in the same vertical plane from a point O at the top of a cliff. The height of the cliff exceeds 50 m. Both particles move freely under gravity. Particle P is projected with speed $\frac{35}{2}\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Particle Q is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle β above the horizontal, where $\tan\beta = \frac{1}{2}$. Particle Q is projected one second after the projection of particle P. The particles collide T s after the projection of particle Q.

(a) Write down expressions, in terms of T, for the horizontal displacements of P and Q from O when they collide and hence show that $4uT = 21\sqrt{5}(T+1)$. [4]

(b) Find the value of T. [4]

(c) Find the horizontal and vertical displacements of the particles from O when they collide. [3]

16. [9231/s22/33/q3]

A particle P is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. After $2\,\mathrm{s}$ the speed of P is $15\,\mathrm{m\,s^{-1}}$.

(a) Find the value of $\sin\theta$. [5]

(b) Find the range of the flight. [3]

17. [9231/w22/31/q7]

A particle P is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle $75°$ above the horizontal from a point O on a horizontal plane. It then moves freely under gravity.

(a) Show that the total time of flight, in seconds, is $\frac{2V}{g}\sin 75°$. [2]

A smooth vertical barrier is now inserted with its lower end on the plane at a distance 15 m from O. The particle is projected as before but now strikes the barrier, rebounds and returns to O. The coefficient of restitution between the barrier and the particle is $\frac{3}{5}$.

(b) Explain why the total time of flight is unchanged. [1]

(c) Find an expression for V in terms of g. [7]

18. [9231/w22/32/q5]

A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle of θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t s are denoted by x m and y m respectively.

(a) Show that the equation of the trajectory is given by

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta).$$ [4]

In the subsequent motion P passes through the point with coordinates (30, 20).

(b) Given that one possible value of $\tan\theta$ is $\frac{4}{3}$, find the other possible value of $\tan\theta$. [5]

19. [9231/s21/31/q7]

A particle P is projected from a point O on a horizontal plane and moves freely under gravity. The initial velocity of P is $100\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. The two times at which P's height above the plane is H m differ by 10 s.

(a) Find the value of H. [5]

(b) Find the magnitude and direction of the velocity of P one second before it strikes the plane. [4]

20. [9231/s21/33/q7]

A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Use the equation of the trajectory given in the List of formulae (MF19), together with the condition $y = 0$, to establish an expression for the range R in terms of u, θ and g. [2]

(b) Deduce an expression for the maximum height H, in terms of u, θ and g. [2]

It is given that $R = \dfrac{4H}{\sqrt{3}}$.

(c) Show that $\theta = 60°$. [1]

It is given also that $u = \sqrt{40}\,\mathrm{m\,s^{-1}}$.

(d) Find, by differentiating the equation of the trajectory or otherwise, the set of values of x for which the direction of motion makes an angle of less than 45° with the horizontal. [4]

21. [9231/w21/31/q5]

A particle P is projected from a point O on a horizontal plane and moves freely under gravity. Its initial speed is $u\,\mathrm{m\,s^{-1}}$ and its angle of projection is $\sin^{-1}(\frac{4}{5})$ above the horizontal. At time 8 s after projection, P is at the point A. At time 32 s after projection, P is at the point B. The direction of motion of P at B is perpendicular to its direction of motion at A.

Find the value of u. [7]

22. [9231/w21/32/q1]

A particle is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane. The particle moves freely under gravity.

(a) Write down the horizontal and vertical components of the velocity of the particle at time T after projection. [2]

At time T after projection, the direction of motion of the particle is perpendicular to the direction of projection.

(b) Express T in terms of u, g and α. [2]

(c) Deduce that $T > \dfrac{u}{g}$. [1]

23. [9231/s20/31/q1]

A particle P is projected with speed u at an angle of 30° above the horizontal from a point O on a horizontal plane and moves freely under gravity. The particle reaches its greatest height at time T after projection.

Find, in terms of u, the speed of P at time $\frac{2}{3}T$ after projection. [5]

24. [9231/s20/33/q6]

A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The direction of motion of P makes an angle α above the horizontal when P first reaches three-quarters of its greatest height.

(a) Show that $\tan\alpha = \frac{1}{2}\tan\theta$. [6]

(b) Given that $\tan\theta = \frac{4}{3}$, find the horizontal distance travelled by P when it first reaches three-quarters of its greatest height. Give your answer in terms of u and g. [4]

25. [9231/w20/31/q5]

A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha.$$ [3]

The point Q is the highest point on the trajectory of P in the case where $\alpha = 45°$.

(b) Show that the x-coordinate of Q is $\frac{u^2}{2g}$. [3]

(c) Find the other value of α for which P would pass through the point Q. [4]

26. [9231/w20/32/q5]

A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle of θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t s are denoted by x m and y m respectively.

(a) Starting from the equation of the trajectory given in the List of formulae (MF19), show that

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta).$$ [1]

When $\theta = \tan^{-1}2$, P passes through the point with coordinates (10, 16).

(b) Show that there is no value of θ for which P can pass through the point with coordinates (18, 30). [6]

27. [9709/m19/52/q1]

A particle is projected with speed $24\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ above the horizontal. Find the speed and direction of motion of the particle at the instant 4 s after projection. [5]

28. [9709/m19/52/q3]

A small ball is projected from a point O on horizontal ground. At time t s after projection the horizontal and vertically upwards displacements of the ball from O are x m and y m respectively, where $x = 4t$ and $y = 6t - 5t^2$.

(i) Find the equation of the trajectory of the ball. [2]

(ii) Hence or otherwise calculate the angle of projection of the ball and its initial speed. [4]

29. [9709/s19/51/q2]

A particle is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal. At the instant 4 s after projection the speed of the particle is $16\,\mathrm{m\,s^{-1}}$ and its direction of motion is 30° above the horizontal. Find V and θ. [5]

30. [9709/s19/51/q4]

A small ball is projected with speed $25\,\text{m}\,\text{s}^{-1}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. At time t s after projection the horizontal and vertically upwards displacements of the ball from O are x m and y m respectively.

(i) Express x and y in terms of t and hence find the equation of the trajectory of the ball. [4]

(ii) Find x for the position of the ball when its path makes an angle of $15°$ below the horizontal. [4]

31. [9709/s19/52/q1]

A small ball is projected from a point O on horizontal ground at an angle of 30° above the horizontal. At time t s after projection the vertically upwards displacement of the ball from O is $(14t - kt^2)$ m, where k is a constant.

(i) State the value of k. [1]

(ii) Show that the initial speed of the ball is $28\,\text{m s}^{-1}$. [2]

(iii) Find the horizontal displacement of the ball from O when $t = 3$. [2]

32. [9709/s19/52/q6]

A particle is projected with speed $15\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal. At the instant 4 s after projection the speed of the particle is $30\,\mathrm{m\,s^{-1}}$.

(i) Find θ. [4]

(ii) Show that at the instant 4 s after projection the particle is 33.75 m below the level of the point of projection and find the direction of motion at this instant. [4]

33. [9709/w19/51/q2]

A particle is projected from a point on horizontal ground with speed $15\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal. The particle strikes the ground 2 s after projection.

(i) Find θ. [2]

(ii) Calculate the time after projection at which the direction of motion of the particle is 20° below the horizontal. [4]

34. [9709/w19/51/q4]

A small ball B is projected with speed $30\,\mathrm{m\,s^{-1}}$ at an angle of 60° above the horizontal from a point O. At time t s after projection the horizontal and vertically upwards displacements of B from O are x m and y m respectively.

(i) Express x and y in terms of t and hence find the equation of the trajectory of the ball. [4]

(ii) Find the value of x for which OB makes an angle of 45° above the horizontal. [3]

35. [9709/w19/52/q2]

A small ball is projected from a point O on horizontal ground at an angle of 30° above the horizontal. At time t s after projection the horizontal and vertically upwards displacements of the ball from O are x m and y m respectively. It is given that $x = 40t$.

(i) Calculate the initial speed of the ball, and express y in terms of t. [3]

(ii) Hence find the equation of the trajectory of the ball. [2]

36. [9709/w19/52/q4]

A particle is projected from a point O on horizontal ground with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of 60° above the horizontal. At the instant 3 s after projection the direction of motion of the particle is 30° below the horizontal.

(i) Find V. [3]

(ii) Calculate the distance of the particle from O at the instant 3 s after projection. [3]

37. [9709/m18/52/q2]

An object is projected with speed $15\,\mathrm{m\,s^{-1}}$ at an angle of $35°$ above the horizontal from a point on horizontal ground. Find the speed and direction of motion of the object at time 2 s after the instant of projection. [5]

38. [9709/m18/52/q4]

A particle P is projected from a point O on horizontal ground. At the instant t s after projection, the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is $y = 3x - 0.05x^2$.

(i) Find the angle of projection and the initial speed of P. [3]

(ii) Find the coordinates of P at the instant when OP makes an angle of 45° with the horizontal. [2]

(iii) For the instant when P is at its greatest height above the ground, calculate this height and the corresponding value of t. [4]

39. [9709/s18/51/q1]

A small ball B is projected from a point O on horizontal ground. The initial velocity of B has horizontal and vertically upwards components of $18\,\mathrm{m\,s^{-1}}$ and $25\,\mathrm{m\,s^{-1}}$ respectively. For the instant 4 s after projection, find the speed and direction of motion of B. [4]

40. [9709/s18/51/q4]

A small object is projected from a point O with speed $V \text{ m s}^{-1}$ at an angle of 45° above the horizontal. At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

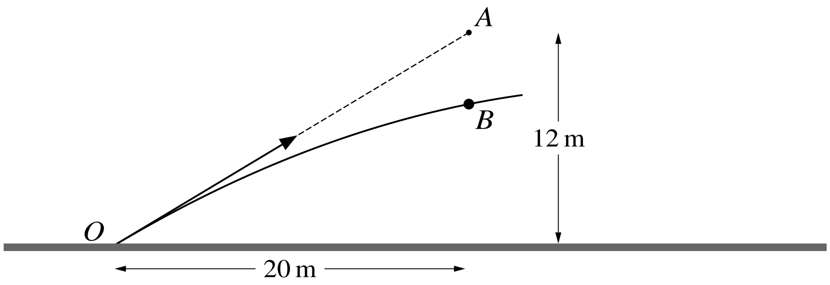
(i) Express x and y in terms of t, and hence find the equation of the path. [4]

The object passes through the point with coordinates (24, 18).

(ii) Find V. [2]

(iii) The object passes through two points which are 22.5 m above the level of O. Find the values of x for these points. [3]

41. [9709/s18/52/q1]



A small ball B is projected from a point O on horizontal ground towards a point A 12 m above the ground. 0.9 s after projection B has travelled a horizontal distance of 20 m and is vertically below A (see diagram).

(i) Find the angle and the speed of projection of B. [4]

(ii) Calculate the distance AB when B is vertically below A. [2]

42. [9709/s18/52/q4]

A particle P is projected from a point O on horizontal ground with initial speed $20\,\mathrm{m\,s^{-1}}$ and angle of projection $30°$. At the instant t s after projection, the horizontal and vertically upwards displacements of P from O are x m and y m respectively.

(i) Express x and y in terms of t and hence find the equation of the trajectory of P. [4]

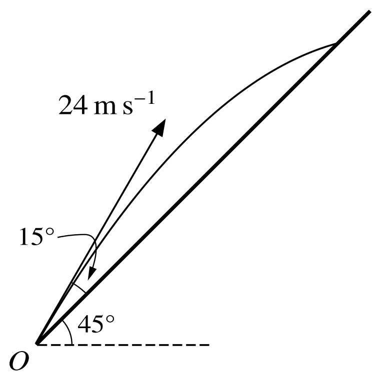
P is at the same height above the ground at two points which are a horizontal distance apart of 15 m.

(ii) Calculate this height. [3]

43. [9709/w18/51/q1]

A small ball B is projected with speed $30\,\mathrm{m\,s^{-1}}$ at an angle of 60° to the horizontal from a point on horizontal ground. Find the time after projection when the speed of B is $25\,\mathrm{m\,s^{-1}}$ for the second time. [4]

44. [9709/w18/51/q7]



A small object is projected with speed $24\,\text{m}\,\text{s}^{-1}$ from a point O at the foot of a plane inclined at 45° to the horizontal. The angle of projection of the object is 15° above a line of greatest slope of the plane (see diagram). At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the value of t for the instant when the object strikes the plane. [4]

(ii) Express the vertical height of the object above the plane in terms of t and hence find the greatest vertical height of the object above the plane. [5]

45. [9709/w18/52/q1]

A small ball B is projected with speed $38\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ to the horizontal from a point on horizontal ground. Find the speed of B when the path of B makes an angle of $20°$ above the horizontal. [3]

46. [9709/w18/52/q4]

A small object is projected horizontally with speed $V\,\text{m}\,\text{s}^{-1}$ from a point O above horizontal ground. At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

(i) Express x and y in terms of t and hence show that the equation of the path of the object is $y = -\dfrac{5x^2}{V^2}$. [3]

The object passes through points with coordinates $(a, -a)$ and $(a^2, -16a)$, where a is a positive constant.

(ii) Find the value of a. [3]

(iii) Given that the object strikes the ground at the point where $x = 5a$, find the height of O above the ground . [2]

47. [9709/m17/52/q1]

A small ball is projected with speed $15\,\mathrm{m\,s^{-1}}$ at an angle of 60° above the horizontal. Find the distance from the point of projection of the ball at the instant when it is travelling horizontally. [5]

48. [9709/m17/52/q3]

A particle P is projected with speed $20\,\mathrm{m\,s^{-1}}$ at an angle of $60°$ below the horizontal, from a point O which is 30 m above horizontal ground.

(i) Calculate the time taken by P to reach the ground. [3]

(ii) Calculate the speed and direction of motion of P immediately before it reaches the ground. [4]

49. [9709/s17/51/q1]

A particle is projected with speed $20\,\text{m}\,\text{s}^{-1}$ at an angle of $60°$ above the horizontal. Calculate the time after projection when the particle is descending at an angle of $40°$ below the horizontal. [4]

50. [9709/s17/51/q4]

A particle is projected from a point O on horizontal ground. The initial components of the velocity of the particle are $10\,\mathrm{m\,s^{-1}}$ horizontally and $15\,\mathrm{m\,s^{-1}}$ vertically. At time t s after projection, the horizontal and vertically upwards displacements of the particle from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the equation of the trajectory of the particle. [4]

The horizontal ground is at the top of a vertical cliff. The point O is at a distance d m from the edge of the cliff. The particle is projected towards the edge of the cliff and does not strike the ground before it passes over the edge of the cliff.

(ii) Show that d is less than 30. [2]

(iii) Find the value of x when the particle is 14 m below the level of O. [2]

51. [9709/s17/52/q7]

A particle P is projected from a point O with speed $V\,\mathrm{m\,s^{-1}}$. At time t s after projection the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is $y = 2x - \dfrac{25x^2}{V^2}$.

(i) Write down the value of $\tan\theta$, where θ is the angle of projection of P. [1]

When $t = 4$, P passes through the point A where $x = y = a$.

(ii) Calculate V and a. [5]

(iii) Find the direction of motion of P when it passes through A. [3]

52. [9709/w17/51/q2]

A small ball is projected from a point 1.5 m above horizontal ground. At a point 9 m above the ground the ball is travelling at 45° above the horizontal and its velocity is $4\ \mathrm{m\,s^{-1}}$. Find the angle of projection of the ball. [4]

53. [9709/w17/51/q4]

A particle P is projected with speed $25\,\text{m}\,\text{s}^{-1}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. At time t s after projection the horizontal and vertically upwards displacements of P from O are x m and y m respectively.

(i) Express x and y in terms of t and hence show that the equation of the trajectory of P is

$$y = \frac{x}{\sqrt{3}} - \frac{4x^2}{375}.$$ [4]

(ii) Find the horizontal distance between the two points at which P is 5 m above the ground. [3]

54. [9709/w17/52/q7]

A small ball B is projected from a point O which is h m above a horizontal plane. At time 2 s after projection B has speed $18\,\text{m}\,\text{s}^{-1}$ and is moving in the direction $30°$ above the horizontal.

(i) Find the initial speed and the angle of projection of B. [4]

B has speed $38\,\text{m}\,\text{s}^{-1}$ immediately before it strikes the plane.

(ii) Calculate h. [2]

B bounces when it strikes the plane, and leaves the plane with speed $20\,\text{m}\,\text{s}^{-1}$ but with its horizontal component of velocity unchanged.

(iii) Find the total time which elapses between the initial projection of B and the instant when it strikes the plane for the second time. [5]

55. [9709/m16/52/q1]

A particle is projected from a point on horizontal ground. At the instant 2 s after projection, the particle has travelled a horizontal distance of 30 m and is at its greatest height above the ground. Find the initial speed and the angle of projection of the particle. [5]

56. [9709/m16/52/q3]

A stone is thrown with speed $9\ \mathrm{m\,s^{-1}}$ at an angle of $60°$ above the horizontal from a point on horizontal ground. Find the distance between the two points at which the path of the stone makes an angle of $45°$ with the horizontal. [5]

57. [9709/s16/51/q1]

A small ball is projected with speed $16\,\mathrm{m\,s^{-1}}$ at an angle of $45°$ above the horizontal from a point on horizontal ground. Calculate the period of time, before the ball lands, for which the speed of the ball is less than $12\,\mathrm{m\,s^{-1}}$. [4]

58. [9709/s16/51/q5]

A particle is projected at an angle of $\theta°$ below the horizontal from a point at the top of a vertical cliff 26 m high. The particle strikes horizontal ground at a distance 8 m from the foot of the cliff 2 s after the instant of projection. Find

(i) the speed of projection of the particle and the value of θ, [6]

(ii) the direction of motion of the particle immediately before it strikes the ground. [3]

59. [9709/s16/52/q1]

A small ball B is projected with speed $12\,\text{m}\,\text{s}^{-1}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. At the instant 0.8 s after projection, B is 0.5 m vertically above the top of a vertical post.

(i) Calculate the height of the top of the post above the ground. [3]

(ii) Show that B is at its greatest height 0.2 s before passing over the post. [2]

60. [9709/s16/52/q3]

The point O is 8 m above a horizontal plane. A particle P is projected from O. After projection, the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is

$$y = 2x - x^2.$$

(i) Find the value of x for the point where P strikes the plane. [2]

(ii) Find the angle and speed of projection of P. [3]

(iii) Calculate the speed of P immediately before it strikes the plane. [2]

61. [9709/w16/51/q7]

A particle P is projected with speed $35\,\mathrm{m\,s^{-1}}$ from a point O on a horizontal plane. In the subsequent motion, the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is

$$y = kx - \frac{(1+k^2)x^2}{245},$$

where k is a constant. P passes through the points $A\,(14,\ a)$ and $B\,(42,\ 2a)$, where a is a constant.

(i) Calculate the two possible values of k and hence show that the larger of the two possible angles of projection is $63.435°$, correct to 3 decimal places. [5]

For the larger angle of projection, calculate

(ii) the time after projection when P passes through A, [2]

(iii) the speed and direction of motion of P when it passes through B. [4]

62. [9709/w16/52/q1]

A stone S is thrown horizontally from the top T of a high tower. At the instant 1.6 s after S is thrown, the line ST makes an angle of 30° below the horizontal. Find the speed with which S is thrown. [3]

63. [9709/w16/52/q4]

A particle P is projected with speed $20\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. P subsequently bounces when it first strikes the ground at the point A.

(i) Find the time after projection when P first strikes the ground, and the distance OA. [3]

When P bounces at A the horizontal component of the velocity of P is unchanged. The vertical component of velocity is $8\,\mathrm{m\,s^{-1}}$ immediately after bouncing. P strikes the ground for the second time at B where it remains at rest.

(ii) Calculate the first and last times after projection at which the speed of P is $18\,\mathrm{m\,s^{-1}}$. [5]

64. [9709/s15/51/q2]

A stone is projected from a point O on horizontal ground. The equation of the trajectory of the stone is

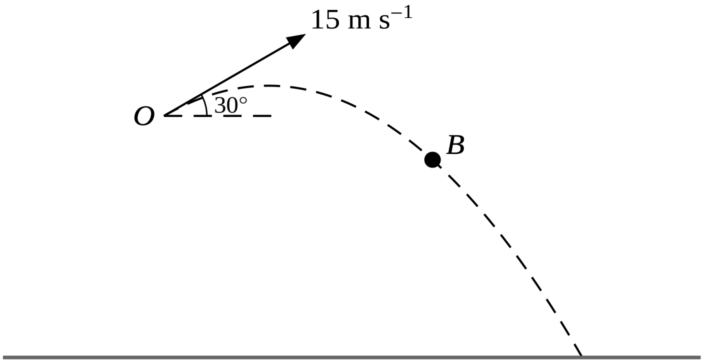
$$y = 1.2x - 0.15x^2,$$

where x m and y m are respectively the horizontal and vertically upwards displacements of the stone from O. Find

(i) the greatest height of the stone, [2]

(ii) the distance from O of the point where the stone strikes the ground. [2]

65. [9709/s15/51/q4]



A small ball B is projected from a point O above horizontal ground, with initial speed $15\,\mathrm{m\,s^{-1}}$ at an angle of projection of 30° above the horizontal (see diagram). The ball strikes the ground 3 s after projection.

(i) Calculate the speed and direction of motion of the ball immediately before it strikes the ground. [5]

(ii) Find the height of O above the ground. [2]

66. [9709/s15/52/q1]

A particle P of mass 0.6 kg is on the rough surface of a horizontal disc with centre O. The distance OP is 0.4 m. The disc and P rotate with angular speed 3 rad s^{-1} about a vertical axis which passes through O. Find the magnitude of the frictional force which the disc exerts on the particle, and state the direction of this force. [3]

67. [9709/s15/52/q6]

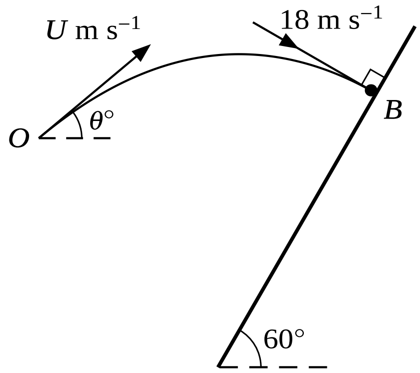


Fig. 1

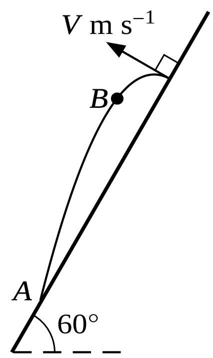


Fig. 2

A small ball B is projected with speed $U\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal from a point O. At time 2 s after the instant of projection, B strikes a smooth wall which slopes at 60° to the horizontal. The speed of B is $18\,\mathrm{m\,s^{-1}}$ and its direction of motion is perpendicular to the wall at the instant of impact (see Fig. 1). B bounces off the wall with speed $V\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to the wall. At time 0.8 s after B bounces off the wall, B strikes the wall again at a lower point A (see Fig. 2).

(i) Find U and θ. [5]

(ii) By considering the motion of B after it bounces off the wall, calculate V. [4]

68. [9709/s15/53/q2]

A particle P is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of $60°$ above the horizontal from a point O on horizontal ground. P is moving at an angle of $45°$ above the horizontal at the instant 1.5 s after projection.

(i) Find V. [3]

(ii) Hence calculate the horizontal and vertical displacements of P from O at the instant 1.5 s after projection. [2]

69. [9709/s15/53/q4]

A small ball B is projected from a point 1.5 m above horizontal ground with initial speed $29\,\mathrm{m\,s^{-1}}$ at an angle of 30° above the horizontal.

(i) Show that B strikes the ground 3 s after projection. [2]

(ii) Find the speed and direction of motion of B immediately before it strikes the ground. [4]

70. [9709/w15/51/q7]

A particle P is projected with speed $V \mathrm{~m~s^{-1}}$ at an angle of $60°$ above the horizontal from a point O. At the instant 1 s later a particle Q is projected from O with the same initial speed at an angle of $45°$ above the horizontal. The two particles collide when Q has been in motion for t s.

(i) Show that $t = 2.414$, correct to 3 decimal places. [3]

(ii) Find the value of V. [4]

The collision occurs after P has passed through the highest point of its trajectory.

(iii) Calculate the vertical distance of P below its greatest height when P and Q collide. [4]

71. [9709/w15/53/q1]

A particle is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle of $50°$ above the horizontal. Calculate the time after projection when the particle has speed $18\,\mathrm{m\,s^{-1}}$ and is rising. [4]

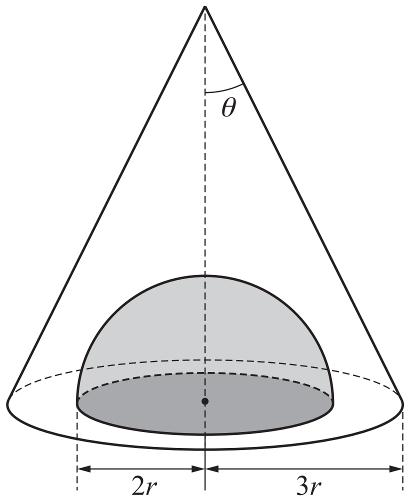
72. [9709/w15/53/q3]

A particle P is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal from a point O on horizontal ground. At the instant 4 s after projection the particle passes through the point A, where $OA = 40$ m and the line OA makes an angle of 30° with the horizontal. Calculate V and θ. [5]

Chapter 2

Equilibrium of a rigid body

1. [9231/s25/31/q4]



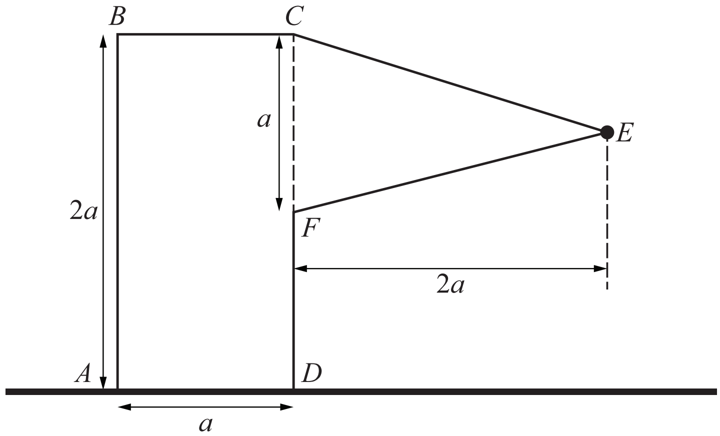
An object is formed by removing a solid hemisphere, radius $2r$, from a uniform solid cone, radius $3r$ and semi-vertical angle θ, where $\tan\theta = \frac{1}{2}$. The axes of symmetry of the cone and the hemisphere coincide. The base of the cone and the base of the hemisphere are in the same plane as each other (see diagram).

(a) Find, in terms of r, the distance of the centre of mass of the object from its base. [4]

The object is placed such that its circular base makes contact with a rough plane which is inclined to the horizontal at an angle α. The object is on the point of toppling. The plane is sufficiently rough to prevent sliding.

(b) Find the value of α. [3]

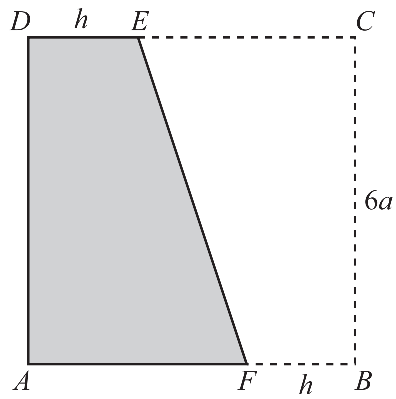
2. [9231/s25/33/q4]



An object consists of a uniform lamina with a particle attached. The uniform lamina *ABCEFD* of mass m is formed from a rectangle *ABCD* and an isosceles triangle *CEF*, where F is the midpoint of *CD*. The rectangle has sides $AB = 2a$ and $AD = a$. The triangle *CEF* has base a and height $2a$. The particle of mass km is attached to the lamina at E. The object rests in a vertical plane with its edge *AD* on horizontal ground (see diagram).

Given that the object is on the point of toppling in its vertical plane about the vertex D, find the value of k. [4]

3. [9231/s25/34/q5]



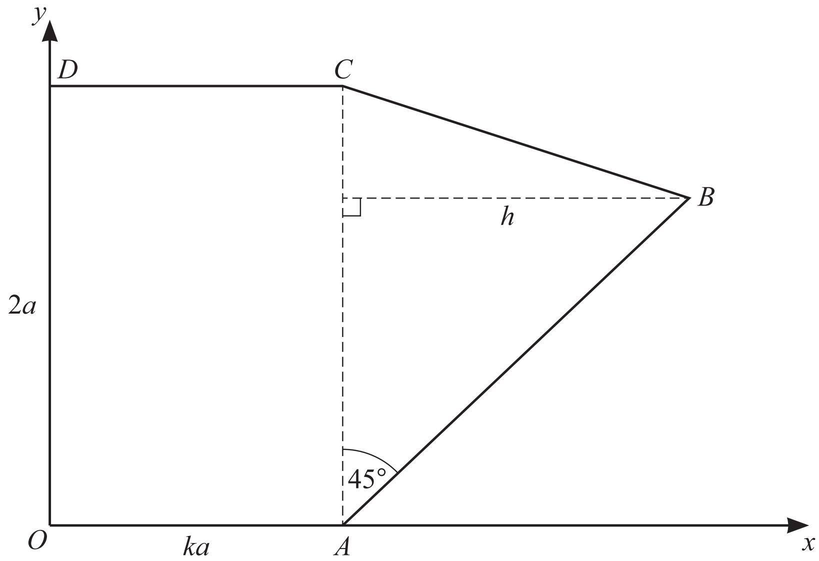
ABCD is a uniform square lamina of side $6a$. Points E and F are on DC and AB respectively and are such that $DE = FB = h$. The quadrilateral *BCEF* is removed from the square lamina (see diagram).

(a) Show that the distance of the centre of mass of the resulting lamina *AFED* from *AD* is $\dfrac{h^2 - 6ah + 36a^2}{18a}$ and find a corresponding expression for the distance of the centre of mass from *AB*. [5]

When the lamina *AFED* is suspended from the point D, the edge DA makes an angle θ with the downward vertical, where $\tan\theta = \frac{7}{15}$.

(b) Find, in terms of a, the two possible values of h. [3]

4. [9231/w25/31/q5]



A uniform lamina *OABCD* consists of a rectangle *OACD* and a triangle *ABC*. The length of *OA* is ka, the length of *OD* is $2a$, the height of triangle *ABC* is h and angle *CAB* is 45° (see diagram). Relative to axes through *O*, parallel and perpendicular to *OA* as shown, the centre of mass of triangle *ABC* is $(\overline{x}, \overline{y})$.

(a) Show that $\overline{x}$ is $\frac{1}{3}(3ka+h)$, and find an expression for $\overline{y}$. [3]

The lamina *OABCD* is placed vertically on its edge *OA* on a horizontal plane.

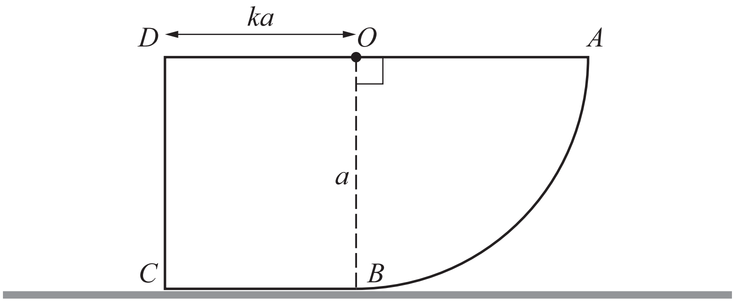
(b) Find, in terms of a and k, the set of values of h for which the lamina is in equilibrium. [4]

It is now given that $k = \frac{\sqrt{3}}{3}$ and that the lamina is on the point of toppling.

(c) Find, in terms of a, the coordinates of the centre of mass of the triangle *ABC*. [2]

5. [9231/w25/32/q3]

A uniform lamina *OABCD* is in the form of a rectangle, *OBCD*, joined along the edge *OB* to a quarter circle *OAB*. The length of *DO* is ka and the length of *OB* is a. The lamina rests in a vertical plane with its edge *CB* on a horizontal surface (see diagram).



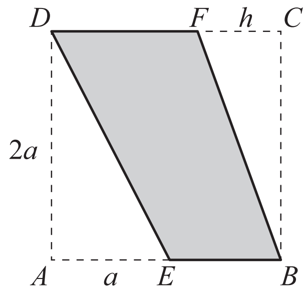
(a) Find, in terms of k, a and π, an expression for the distance of the centre of mass above the horizontal surface. [4]

[You may use without proof the result for the centre of mass of a circular sector in the list of formulae (MF19).]

The lamina is on the point of toppling about *B*.

(b) Find the value of k. [2]

6. [9231/w25/34/q3]



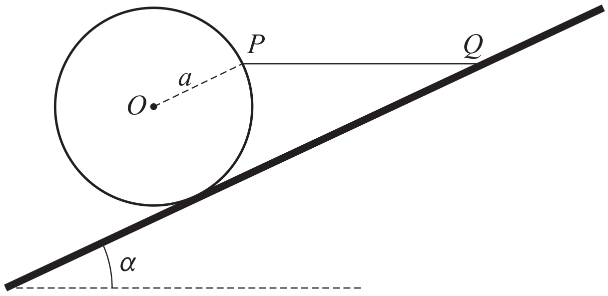
The lamina *BFDE* is obtained by removing triangles *AED* and *BCF* from a uniform square lamina *ABCD* of side $2a$. The length of side *AE* is a and the length of side *FC* is h (see diagram). The centre of mass of *BFDE* is at a distance $\overline{x}$ from *AD*, and at a distance $\overline{y}$ from *AB*.

(a) Show that $\overline{x} = \dfrac{h^2 - 6ah + 11a^2}{3(3a-h)}$ and find a corresponding expression for $\overline{y}$. [5]

(b) The lamina *BFDE* is placed vertically on its edge *EB* on a smooth horizontal surface.

Find, in terms of a, the set of possible values of h for which the lamina remains in equilibrium. [2]

7. [9231/s24/31/q4]



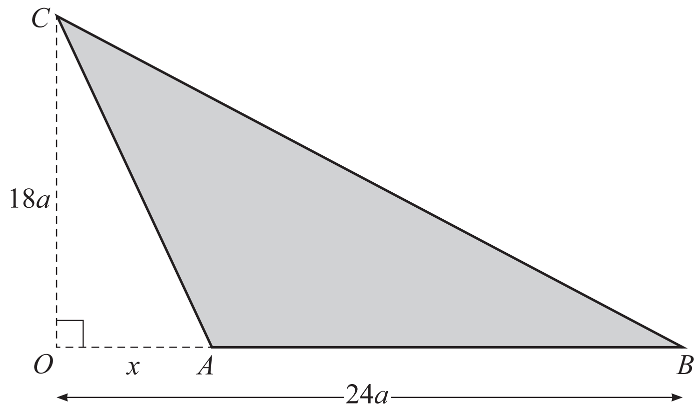
A ring of weight W, with radius a and centre O, is at rest on a rough surface that is inclined to the horizontal at an angle α where $\tan\alpha = \frac{1}{2}$. The plane of the ring is perpendicular to the inclined surface and parallel to a line of greatest slope of the surface. The point P on the circumference of the ring is such that OP is parallel to the surface.

A light inextensible string is attached to P and to the point Q, which is on the surface, such that PQ is horizontal (see diagram). The points O, P and Q are in the same vertical plane. The system is in limiting equilibrium and the coefficient of friction between the ring and the surface is μ.

(a) Find, in terms of W, the tension in the string PQ. [4]

(b) Find the value of μ. [3]

8. [9231/s24/33/q5]



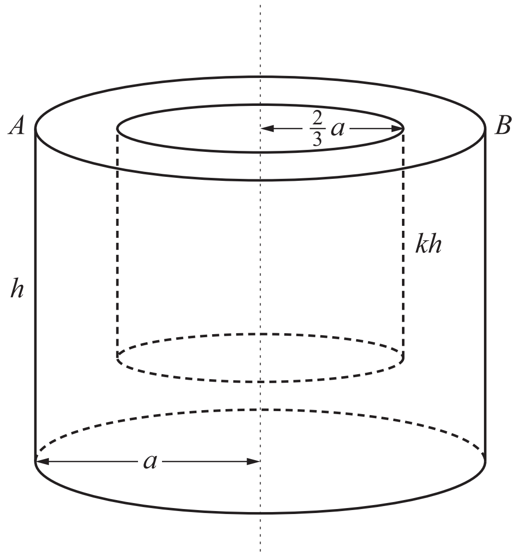
A uniform lamina is in the form of a triangle OBC, with $OC = 18a$, $OB = 24a$ and angle $COB = 90°$. The point A on OB is such that $OA = x$ (see diagram). The triangle OAC is removed from the lamina.

(a) Find, in terms of a and x, the distance of the centre of mass of the remaining object ABC from OC. [3]

The object ABC is suspended from C. In its equilibrium position, the side AB makes an angle θ with the vertical, where $\tan\theta = \frac{6}{5}$.

(b) Find x in terms of a. [4]

9. [9231/w24/31/q4]



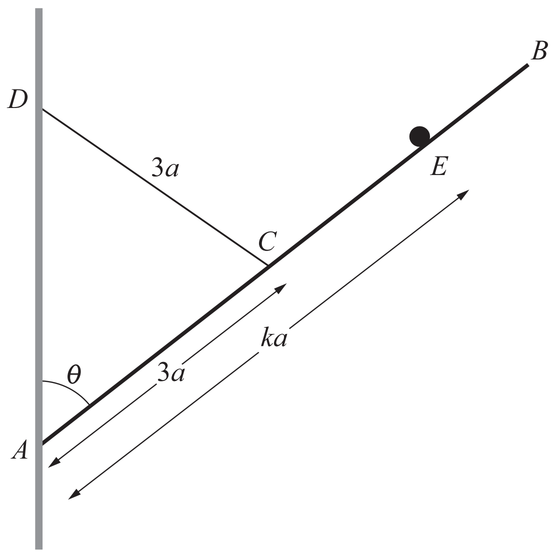
An object is formed by removing a cylinder of radius $\frac{2}{3}a$ and height kh ($k < 1$) from a uniform solid cylinder of radius a and height h. The vertical axes of symmetry of the two cylinders coincide. The upper faces of the two cylinders are in the same plane as each other. The points A and B are the opposite ends of a diameter of the upper face of the object (see diagram).

(a) Find, in terms of h and k, the distance of the centre of mass of the object from AB. [4]

When the object is suspended from A, the angle between AB and the vertical is θ, where $\tan\theta = \frac{3}{2}$.

(b) Given that $h = \frac{8}{3}a$, find the possible values of k. [3]

10. [9231/w24/32/q4]

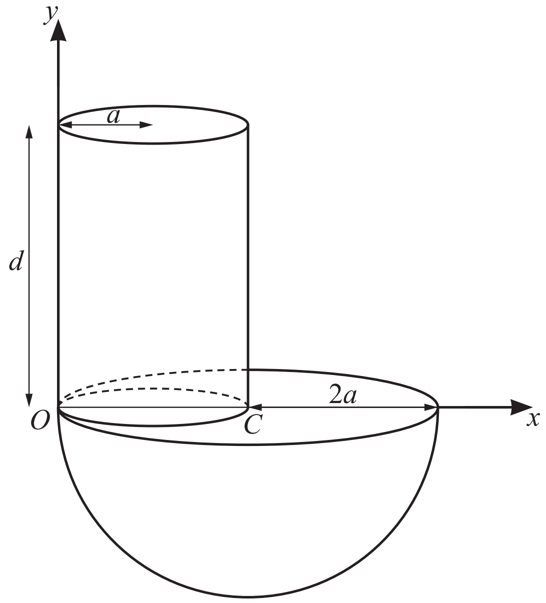


The end A of a uniform rod AB of length $6a$ and weight W is in contact with a rough vertical wall. One end of a light inextensible string of length $3a$ is attached to the midpoint C of the rod. The other end of the string is attached to a point D on the wall, vertically above A. The rod is in equilibrium when the angle between the rod and the wall is θ, where $\tan\theta = \frac{3}{2}$. A particle of weight W is attached to the point E on the rod, where the distance AE is equal to ka $(3 < k < 6)$ (see diagram). The rod and the string are in a vertical plane perpendicular to the wall. The coefficient of friction between the rod and the wall is $\frac{1}{3}$. The rod is about to slip down the wall.

(a) Find the value of k. [5]

(b) Find, in terms of W, the magnitude of the frictional force between the rod and the wall. [2]

11. [9231/s23/31/q4]



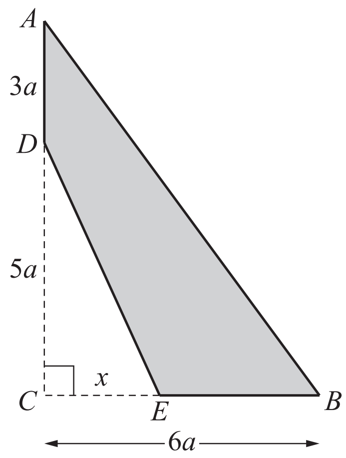
An object is formed from a solid hemisphere, of radius $2a$, and a solid cylinder, of radius a and height d. The hemisphere and the cylinder are made of the same material. The cylinder is attached to the plane face of the hemisphere. The line OC forms a diameter of the base of the cylinder, where C is the centre of the plane face of the hemisphere and O is common to both circumferences (see diagram). Relative to axes through O, parallel and perpendicular to OC as shown, the centre of mass of the object is $(\overline{x}, \overline{y})$.

(a) Show that $\overline{x} = \dfrac{32a^2 + 3ad}{16a + 3d}$ and find an expression, in terms of a and d, for $\overline{y}$. [5]

The object is placed on a rough plane which is inclined to the horizontal at an angle θ where $\sin\theta = \frac{1}{6}$. The object is in equilibrium with CO horizontal, where CO lies in a vertical plane through a line of greatest slope.

(b) Find d in terms of a. [3]

12. [9231/s23/33/q3]



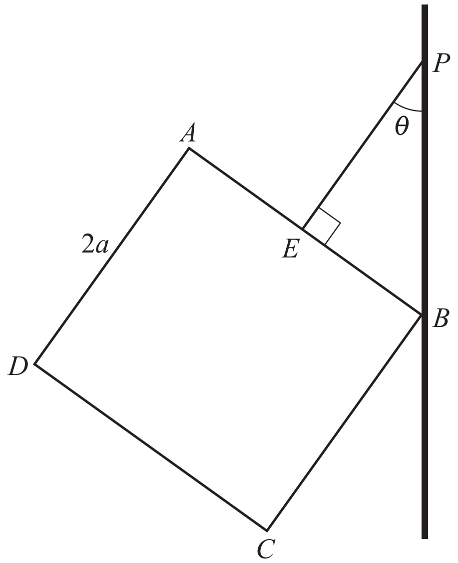
A uniform lamina is in the form of a triangle ABC, with $AC = 8a$, $BC = 6a$ and angle $ACB = 90°$. The point D on AC is such that $AD = 3a$. The point E on CB is such that $CE = x$ (see diagram). The triangle CDE is removed from the lamina.

(a) Find, in terms of a and x, the distance of the centre of mass of the remaining object $ADEB$ from AC. [4]

The object $ADEB$ is on the point of toppling about the point E when the object is in the vertical plane with its edge EB on a smooth horizontal surface.

(b) Find x in terms of a. [3]

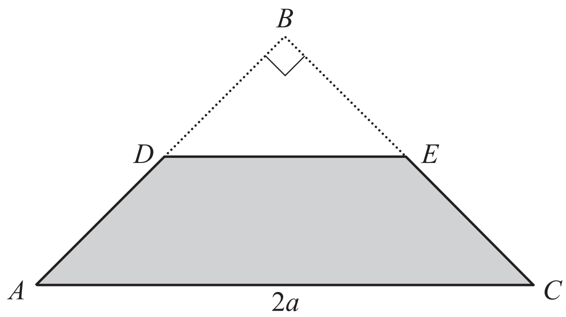
13. [9231/w23/31/q3]



A uniform square lamina of side $2a$ and weight W is suspended from a light inextensible string attached to the midpoint E of the side AB. The other end of the string is attached to a fixed point P on a rough vertical wall. The vertex B of the lamina is in contact with the wall. The string EP is perpendicular to the side AB and makes an angle θ with the wall (see diagram). The string and the lamina are in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the lamina is $\frac{1}{2}$.

Given that the vertex B is about to slip up the wall, find the value of $\tan\theta$. [8]

14. [9231/w23/32/q3]



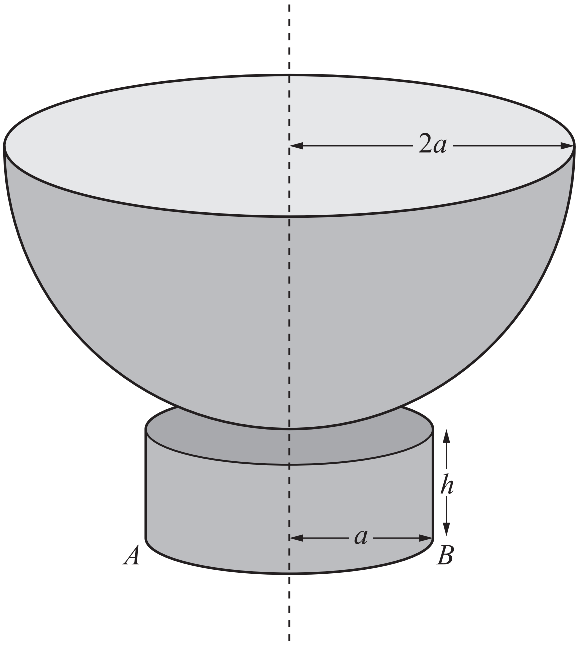
A uniform lamina is in the form of an isosceles triangle ABC in which $AC = 2a$ and angle $ABC = 90°$. The point D on AB is such that the ratio $DB : AB = 1 : k$. The point E on CB is such that DE is parallel to AC. The triangle DBE is removed from the lamina (see diagram).

(a) Find, in terms of k, the distance of the centre of mass of the remaining lamina $ADEC$ from the midpoint of AC. [4]

When the lamina $ADEC$ is freely suspended from the vertex A, the edge AC makes an angle θ with the downward vertical, where $\tan\theta = \frac{5}{18}$.

(b) Find the value of k. [3]

15. [9231/s22/31/q4]



An object is composed of a hemispherical shell of radius $2a$ attached to a closed hollow circular cylinder of height h and base radius a. The hemispherical shell and the hollow cylinder are made of the same uniform material. The axes of symmetry of the shell and the cylinder coincide. AB is a diameter of the lower end of the cylinder (see diagram).

(a) Find, in terms of a and h, an expression for the distance of the centre of mass of the object from AB. [4]

The object is placed on a rough plane which is inclined to the horizontal at an angle θ, where $\tan\theta = \frac{2}{3}$. The object is in equilibrium with AB in contact with the plane and lying along a line of greatest slope of the plane.

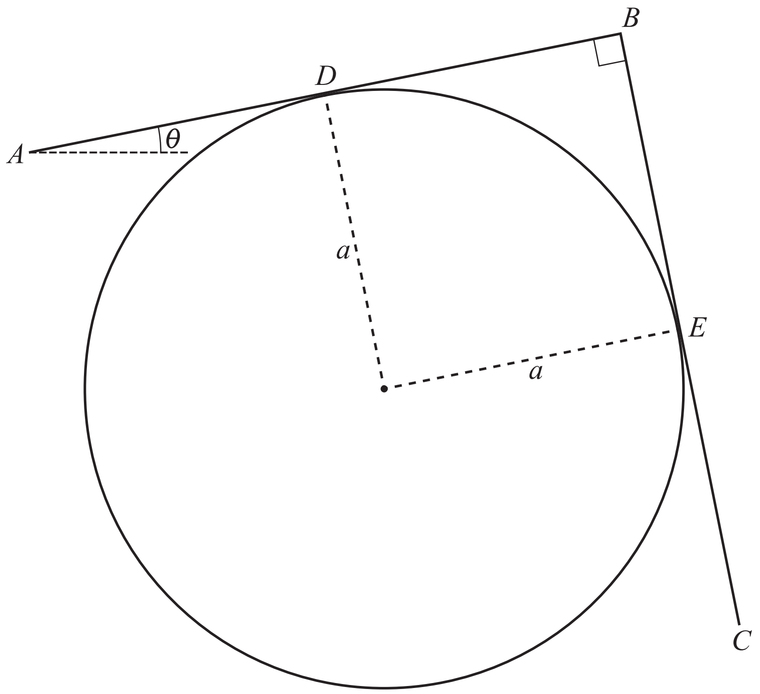
(b) Find the set of possible values of h, in terms of a. [4]

16. [9231/s22/33/q1]

A uniform lamina $OABC$ is a trapezium whose vertices can be represented by coordinates in the x-y plane. The coordinates of the vertices are $O(0,0)$, $A(15,0)$, $B(9,4)$ and $C(3,4)$.

Find the x-coordinate of the centre of mass of the lamina. [4]

17. [9231/s22/33/q7]



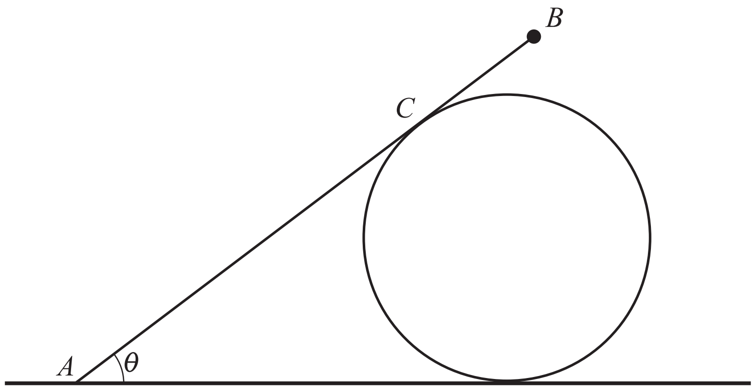
A uniform cylinder with a rough surface and of radius a is fixed with its axis horizontal. Two identical uniform rods AB and BC, each of weight W and length $2a$, are rigidly joined at B with AB perpendicular to BC. The rods rest on the cylinder in a vertical plane perpendicular to the axis of the cylinder with AB at an angle θ to the horizontal. D and E are the midpoints of AB and BC respectively and also the points of contact of the rods with the cylinder (see diagram). The rods are about to slip in a clockwise direction. The coefficient of friction between each rod and the cylinder is μ.

The normal reaction between AB and the cylinder is R and the normal reaction between BC and the cylinder is N.

(a) Find the ratio $R : N$ in terms of μ. [6]

(b) Given that $\mu = \frac{1}{3}$, find the value of $\tan\theta$. [3]

18. [9231/w22/31/q3]



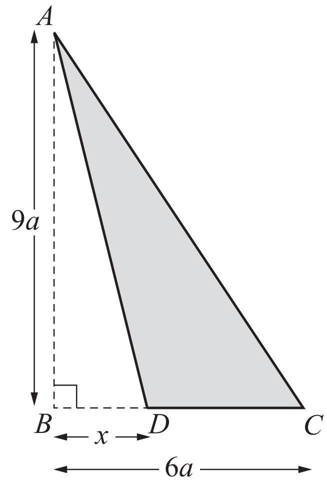
A smooth cylinder is fixed to a rough horizontal surface with its axis of symmetry horizontal. A uniform rod AB, of length $4a$ and weight W, rests against the surface of the cylinder. The end A of the rod is in contact with the horizontal surface. The vertical plane containing the rod AB is perpendicular to the axis of the cylinder. The point of contact between the rod and the cylinder is C, where $AC = 3a$. The angle between the rod and the horizontal surface is θ where $\tan\theta = \frac{3}{4}$ (see diagram). The coefficient of friction between the rod and the horizontal surface is $\frac{6}{7}$.

A particle of weight kW is attached to the rod at B. The rod is about to slip. The normal reaction between the rod and the cylinder is N.

(a) Show that $N = \frac{8}{15}W(1+2k)$. [2]

(b) Find the value of k. [5]

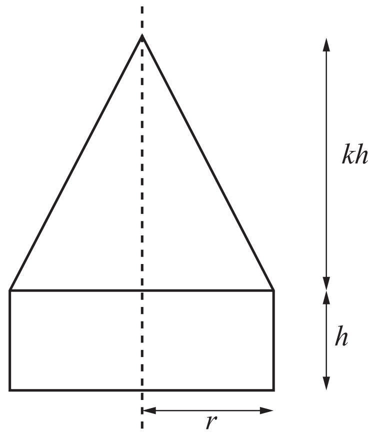
19. [9231/w22/32/q2]



A uniform lamina is in the form of a triangle ABC in which angle B is a right angle, $AB = 9a$ and $BC = 6a$. The point D is on BC such that $BD = x$ (see diagram). The region ABD is removed from the lamina. The resulting shape ADC is placed with the edge DC on a horizontal surface and the plane ADC is vertical.

Find the set of values of x, in terms of a, for which the shape is in equilibrium. [6]

20. [9231/s21/31/q4]



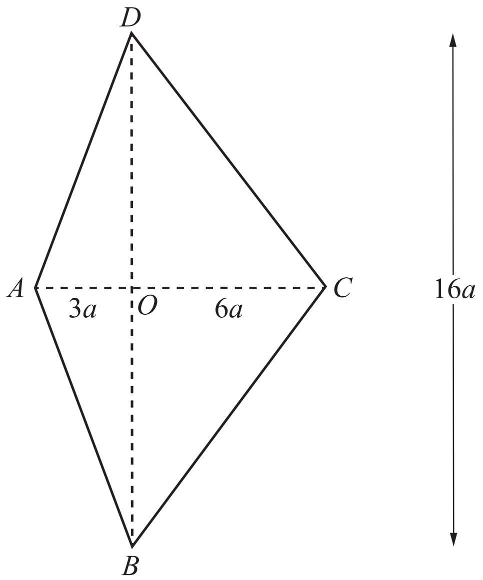
A uniform solid circular cone has vertical height kh and radius r. A uniform solid cylinder has height h and radius r. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram, which shows a cross-section). The cone and the cylinder are made of the same material.

(a) Show that the distance of the centre of mass of the combined solid from the base of the cylinder is $\dfrac{h(k^2+4k+6)}{4(3+k)}$. [4]

The solid is placed on a plane that is inclined to the horizontal at an angle θ. The base of the cylinder is in contact with the plane. The plane is sufficiently rough to prevent sliding. It is given that $3h = 2r$ and that the solid is on the point of toppling when $\tan\theta = \frac{4}{3}$.

(b) Find the value of k. [3]

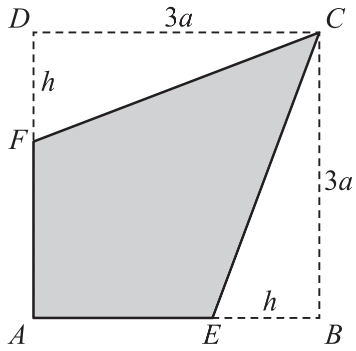
21. [9231/s21/33/q1]



A uniform lamina *ABCD* consists of two isosceles triangles *ABD* and *BCD*. The diagonals of *ABCD* meet at the point *O*. The length of *AO* is $3a$, the length of *OC* is $6a$ and the length of *BD* is $16a$ (see diagram).

Find the distance of the centre of mass of the lamina from *DB*. [3]

22. [9231/w21/31/q4]



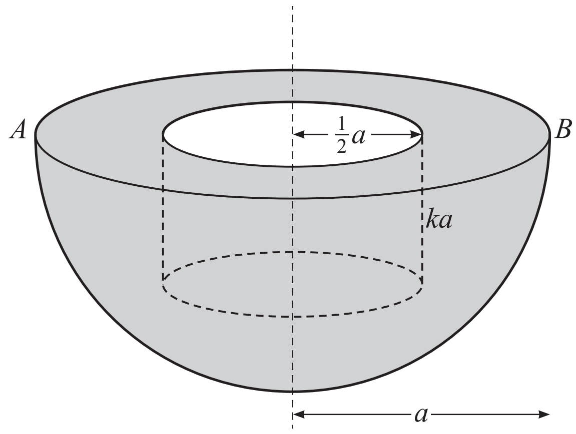
A uniform lamina *AECF* is formed by removing two identical triangles *BCE* and *CDF* from a square lamina *ABCD*. The square has side $3a$ and $EB = DF = h$ (see diagram).

(a) Find the distance of the centre of mass of the lamina *AECF* from *AD* and from *AB*, giving your answers in terms of a and h. [5]

The lamina *AECF* is placed vertically on its edge *AE* on a horizontal plane.

(b) Find, in terms of a, the set of values of h for which the lamina remains in equilibrium. [3]

23. [9231/w21/32/q4]



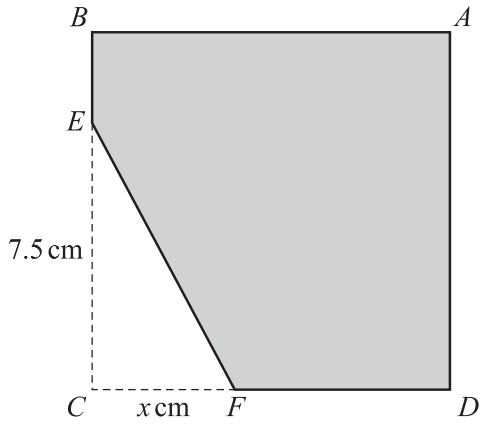
An object is formed by removing a solid cylinder, of height ka and radius $\frac{1}{2}a$, from a uniform solid hemisphere of radius a. The axes of symmetry of the hemisphere and the cylinder coincide and one circular face of the cylinder coincides with the plane face of the hemisphere. AB is a diameter of the circular face of the hemisphere (see diagram).

(a) Show that the distance of the centre of mass of the object from AB is $\dfrac{3a(2-k^2)}{2(8-3k)}$. [4]

When the object is freely suspended from the point A, the line AB makes an angle θ with the downward vertical, where $\tan\theta = \frac{7}{18}$.

(b) Find the possible values of k. [3]

24. [9231/s20/31/q4]



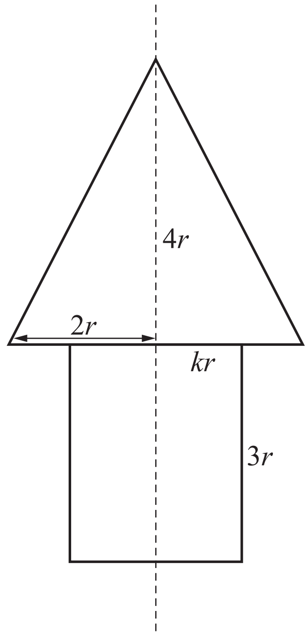
A uniform square lamina *ABCD* has sides of length 10 cm. The point *E* is on *BC* with $EC = 7.5\,\text{cm}$, and the point *F* is on *DC* with $CF = x\,\text{cm}$. The triangle *EFC* is removed from *ABCD* (see diagram). The centre of mass of the resulting shape *ABEFD* is a distance $\bar{x}$ cm from *CB* and a distance $\bar{y}$ cm from *CD*.

(a) Show that $\bar{x} = \dfrac{400 - x^2}{80 - 3x}$ and find a corresponding expression for $\bar{y}$. [4]

The shape *ABEFD* is in equilibrium in a vertical plane with the edge *DF* resting on a smooth horizontal surface.

(b) Find the greatest possible value of *x*, giving your answer in the form $a + b\sqrt{2}$, where *a* and *b* are constants to be determined. [3]

25. [9231/s20/33/q4]



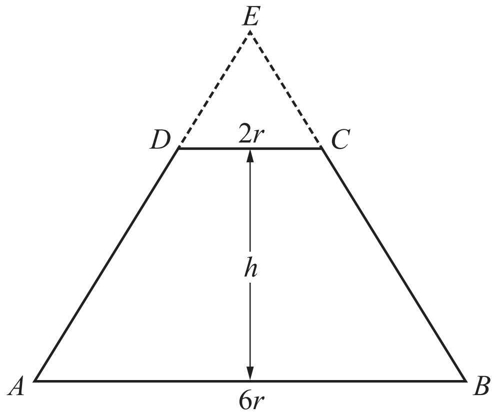
A uniform solid circular cone, of vertical height $4r$ and radius $2r$, is attached to a uniform solid cylinder, of height $3r$ and radius kr, where k is a constant less than 2. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram). The cone and the cylinder are made of the same material.

(a) Show that the distance of the centre of mass of the combined solid from the vertex of the cone is $\dfrac{(99k^2+96)r}{18k^2+32}$. [4]

The point C is on the circumference of the base of the cone. When the combined solid is freely suspended from C and hanging in equilibrium, the diameter through C makes an angle α with the downward vertical, where $\tan\alpha = \frac{1}{8}$.

(b) Given that the centre of mass of the combined solid is within the cylinder, find the value of k. [4]

26. [9231/w20/31/q4]



The diagram shows the cross-section *ABCD* of a uniform solid object which is formed by removing a cone with cross-section *DCE* from the top of a larger cone with cross-section *ABE*. The perpendicular distance between *AB* and *DC* is h, the diameter *AB* is $6r$ and the diameter *DC* is $2r$.

(a) Find an expression, in terms of h, for the distance of the centre of mass of the solid object from *AB*. [4]

The object is freely suspended from the point *B* and hangs in equilibrium. The angle between *AB* and the downward vertical through *B* is θ.

(b) Given that $h = \frac{13}{4}r$, find the value of $\tan\theta$. [2]

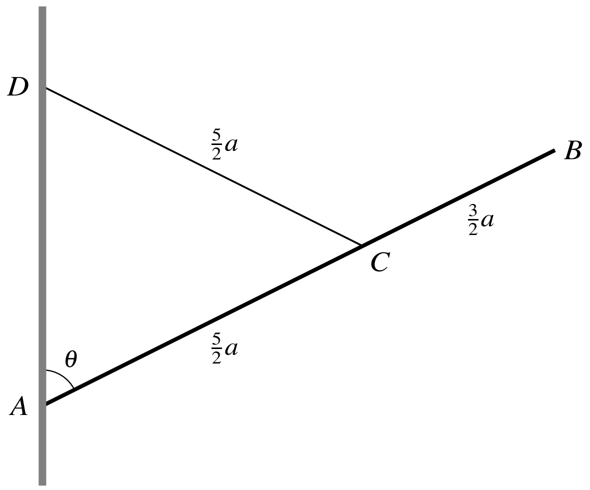
27. [9231/w20/32/q3]

An object consists of a uniform solid circular cone, of vertical height $4r$ and radius $3r$, and a uniform solid cylinder, of height $4r$ and radius $3r$. The circular base of the cone and one of the circular faces of the cylinder are joined together so that they coincide. The cone and the cylinder are made of the same material.

(a) Find the distance of the centre of mass of the object from the end of the cylinder that is not attached to the cone. [4]

(b) Show that the object can rest in equilibrium with the curved surface of the cone in contact with a horizontal surface. [3]

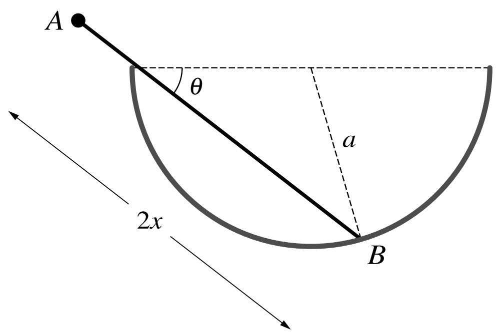
28. [9231/s19/21/q4]



A uniform rod AB of length $4a$ and weight W rests with the end A in contact with a rough vertical wall. A light inextensible string of length $\frac{5}{2}a$ has one end attached to the point C on the rod, where $AC = \frac{5}{2}a$. The other end of the string is attached to a point D on the wall, vertically above A. The vertical plane containing the rod AB is perpendicular to the wall. The angle between the rod and the wall is θ, where $\tan\theta = 2$ (see diagram). The end A of the rod is on the point of slipping down the wall and the coefficient of friction between the rod and the wall is μ.

Find, in either order, the tension in the string and the value of μ. [10]

29. [9231/s19/23/q5]



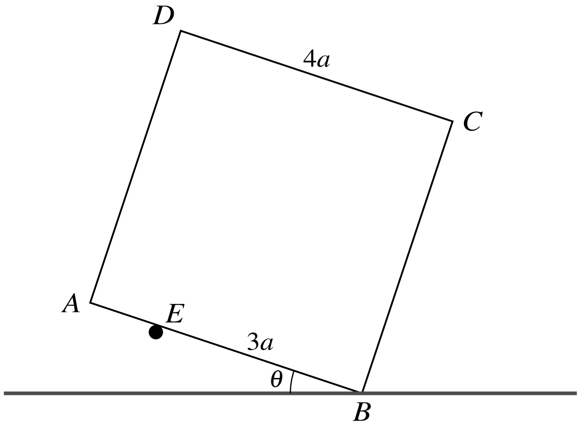
A uniform rod AB of length $2x$ and weight W rests on the smooth rim of a fixed hemispherical bowl of radius a. The end B of the rod is in contact with the rough inner surface of the bowl. The coefficient of friction between the rod and the bowl at B is $\frac{1}{3}$. A particle of weight $\frac{1}{4}W$ is attached to the end A of the rod. The end B is about to slip upwards when AB is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$ (see diagram).

(i) By resolving parallel to the rod, show that the normal component of the reaction of the bowl on the rod at B is $\frac{3}{4}W$. [5]

(ii) Find, in terms of W, the reaction between the rod and the smooth rim of the bowl. [4]

(iii) Find x in terms of a. [3]

30. [9231/w19/21/q2]

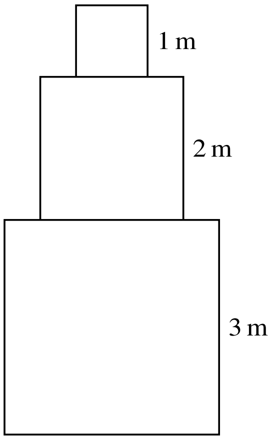


A uniform square lamina $ABCD$ of side $4a$ and weight W rests in a vertical plane with the edge AB inclined at an angle θ to the horizontal, where $\tan\theta = \frac{1}{3}$. The vertex B is in contact with a rough horizontal surface for which the coefficient of friction is μ. The lamina is supported by a smooth peg at the point E on AB, where $BE = 3a$ (see diagram).

(i) Find expressions in terms of W for the normal reaction forces at E and B. [5]

(ii) Given that the lamina is about to slip, find the value of μ. [3]

31. [9709/m19/52/q2]



A uniform object is made by joining together three solid cubes with edges 3 m, 2 m and 1 m. The object has an axis of symmetry, with the cubes stacked vertically and the cube of edge 2 m between the other two cubes (see diagram).

(i) Calculate the distance of the centre of mass of the object above the base of the largest cube. [3]

The smallest cube is now removed from the object. It is replaced by a heavier uniform cube with 1 m edges which is made of a different material. The centre of mass of the object is now at the base of the 2 m cube.

(ii) Find the ratio of the masses of the two cubes of edge 1 m. [3]

32. [9709/m19/52/q6]

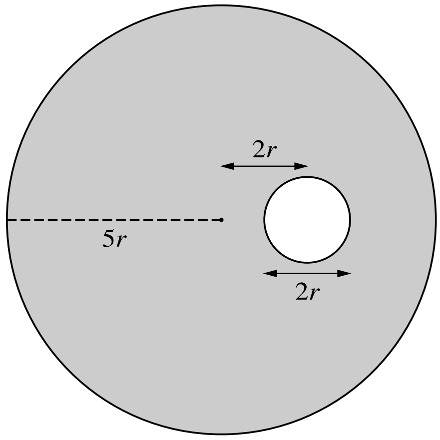


Fig. 1

Fig. 1 shows the cross-section of a solid cylinder through which a cylindrical hole has been drilled to make a uniform prism. The radius of the cylinder is $5r$ and the radius of the hole is r. The centre of the hole is a distance $2r$ from the centre of the cylinder.

(i) Find, in terms of r, the distance of the centre of mass of the prism from the centre of the cylinder. [4]

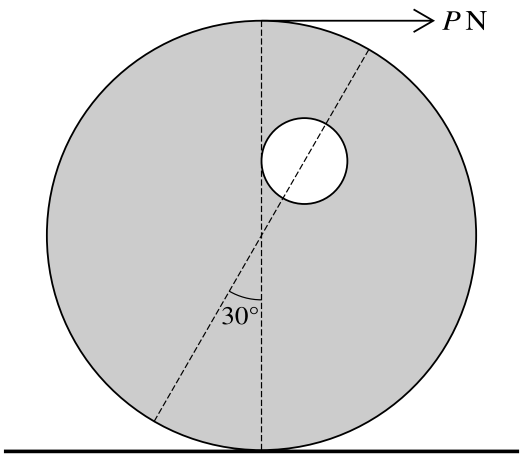
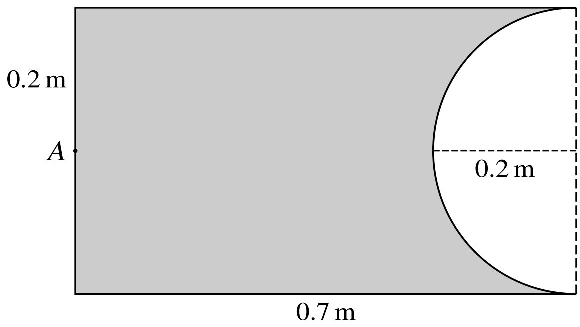


Fig. 2

The prism has weight W N and is placed with its curved surface on a rough horizontal plane. The axis of symmetry of the cross-section makes an angle of 30° with the vertical. A horizontal force of magnitude P N acting in the plane of the cross-section through the centre of mass is applied to the cylinder at the highest point of this cross-section (see Fig. 2). The prism rests in limiting equilibrium.

(ii) Find the coefficient of friction between the prism and the plane. [4]

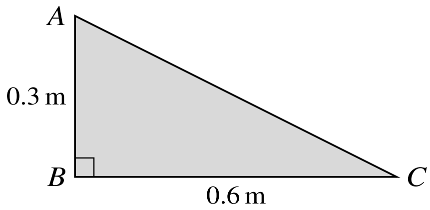
33. [9709/s19/51/q3]



The diagram shows the cross-section through the centre of mass of a uniform solid object. The object is a cylinder of radius 0.2 m and length 0.7 m, from which a hemisphere of radius 0.2 m has been removed at one end. The point A is the centre of the plane face at the other end of the object. Find the distance of the centre of mass of the object from A. [5]

[The volume of a hemisphere is $\frac{2}{3}\pi r^3$.]

34. [9709/s19/51/q6]



ABC is a uniform lamina in the form of a triangle with $AB = 0.3$ m, $BC = 0.6$ m and a right angle at B (see diagram).

(i) State the distances of the centre of mass of the lamina from AB and from BC. [2]

Distance from AB ....

Distance from BC ....

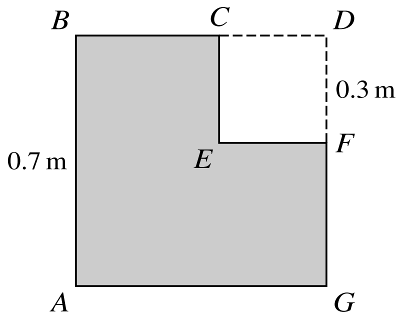
The lamina is freely suspended at B and hangs in equilibrium.

(ii) Find the angle between AB and the horizontal. [2]

A force of magnitude 12 N is applied along the edge AC of the lamina in the direction from A towards C. The lamina, still suspended at B, is now in equilibrium with AB vertical.

(iii) Calculate the weight of the lamina. [3]

35. [9709/s19/52/q2]



A uniform lamina $ABCEFG$ is formed from a square $ABDG$ by removing a smaller square $CDFE$ from one corner. $AB = 0.7$ m and $DF = 0.3$ m (see diagram). Find the distance of the centre of mass of the lamina from A. [4]

36. [9709/s19/52/q7]

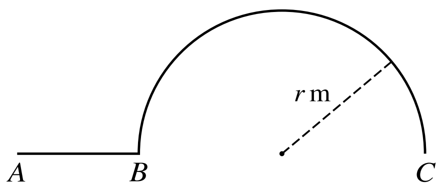


Fig. 1

Fig. 1 shows an object made from a uniform wire of length 0.8 m. The object consists of a straight part AB, and a semicircular part BC such that A, B and C lie in the same straight line. The radius of the semicircle is r m and the centre of mass of the object is 0.1 m from line ABC.

(i) Show that $r = 0.2$. [3]

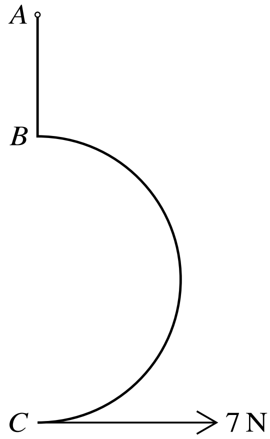


Fig. 2

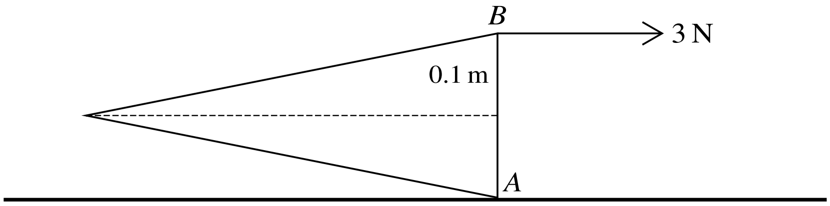
The object is freely suspended at A and a horizontal force of magnitude 7 N is applied to the object at C so that the object is in equilibrium with ABC vertical (see Fig. 2).

(ii) Calculate the weight of the object. [3]

The 7 N force is removed and the object hangs in equilibrium with ABC at an angle of $\theta°$ with the vertical.

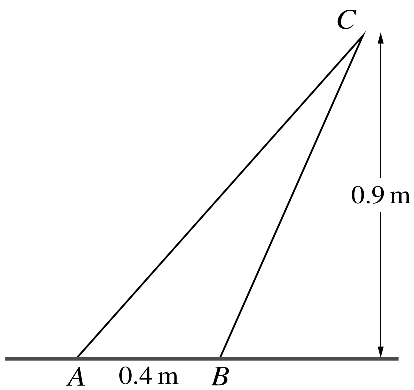
(iii) Find θ. [6]

37. [9709/w19/51/q1]



A uniform solid cone has weight 5 N and base radius 0.1 m. AB is a diameter of the base of the cone. The cone is held in equilibrium, with A in contact with a rough horizontal surface and AB vertical, by a force applied at B. This force has magnitude 3 N and acts parallel to the axis of the cone (see diagram). Calculate the height of the cone. [3]

38. [9709/w19/51/q7]



ABC is the cross-section through the centre of mass of a uniform prism which rests with AB on a rough horizontal surface. $AB = 0.4$ m and C is 0.9 m above the surface (see diagram). The prism is on the point of toppling about its edge through B.

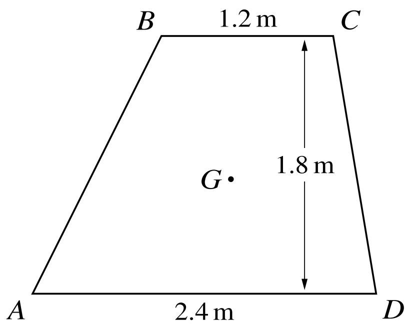
(i) Show that angle $BAC = 48.4°$, correct to 3 significant figures. [3]

A force of magnitude 18 N acting in the plane of the cross-section and perpendicular to AC is now applied to the prism at C. The prism is on the point of rotating about its edge through A.

(ii) Calculate the weight of the prism. [3]

(iii) Given also that the prism is on the point of slipping, calculate the coefficient of friction between the prism and the surface. [4]

39. [9709/w19/52/q7]



ABCD is a uniform lamina in the shape of a trapezium which has centre of mass G. The sides AD and BC are parallel and 1.8 m apart, with $AD = 2.4$ m and $BC = 1.2$ m (see diagram).

(i) Show that the distance of G from AD is 0.8 m. [4]

The lamina is freely suspended at A and hangs in equilibrium with AD making an angle of 30° with the vertical.

(ii) Calculate the distance AG. [2]

With the lamina still freely suspended at A a horizontal force of magnitude 7 N acting in the plane of the lamina is applied at D. The lamina is in equilibrium with AG making an angle of 10° with the downward vertical.

(iii) Find the two possible values for the weight of the lamina. [5]

40. [9231/s18/21/q4]

A uniform rod AB has length $2a$ and weight W. The end A rests on rough horizontal ground and the end B rests against a smooth vertical wall. The rod is in a vertical plane that is perpendicular to the wall. The angle between the rod and the horizontal is θ. A particle of weight $5W$ hangs from the rod at the point C, with $AC = xa$, where $0 < x < 1$.

(i) By taking moments about A, show that the magnitude of the normal reaction at B is $\dfrac{W(5x+1)}{2\tan\theta}$. [3]

The particle of weight $5W$ is now moved a distance a up the rod, so that $AC = (x+1)a$. This results in the magnitude of the normal reaction at B being double its previous value. The system remains in equilibrium with the rod at angle θ with the horizontal.

(ii) Show that $x = \frac{4}{5}$. [3]

The coefficient of friction between the rod and the ground is $\frac{2}{3}$.

(iii) Given that the rod is about to slip when the particle of weight $5W$ is in its second position, find the value of $\tan\theta$. [5]

41. [9231/s18/23/q4]

A uniform rod AB has length $2a$ and weight W. The end A rests on rough horizontal ground and the end B rests against a smooth vertical wall. The angle between the rod and the horizontal is θ, where $\tan\theta = \frac{4}{3}$. One end of a light inextensible rope is attached to a point C on the rod. The other end is attached to a point where the vertical wall and the horizontal ground meet. The rope is taut and perpendicular to the rod. The rope and rod are in a vertical plane perpendicular to the wall.

(i) Show that $AC = \frac{18}{25}a$. [2]

The magnitude of the frictional force at A is equal to one quarter of the magnitude of the normal reaction force at A.

(ii) Show that the tension in the rope is $\frac{1}{4}W$. [6]

(iii) Find expressions, in terms of W, for the magnitudes of the normal reaction forces at A and B. [2]

42. [9231/w18/21/q4]

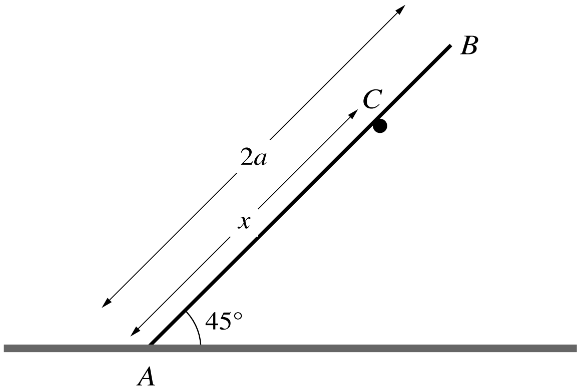
A uniform rod AB of length $4a$ and weight W is smoothly hinged to a vertical wall at the end A. The rod is held at an angle θ above the horizontal by a light elastic string. One end of the string is attached to the point C on the rod, where $AC = 3a$. The other end of the string is attached to a point D on the wall, with D vertically above A and such that angle $ACD = 2\theta$. A particle of weight $\frac{1}{2}W$ is attached to the rod at B. It is given that $\tan\theta = \frac{8}{15}$.

(i) Show that the tension in the string is $\frac{17}{12}W$. [4]

(ii) Find the magnitude and direction of the reaction at the hinge. [5]

(iii) Given that the natural length of the string is $2a$, find its modulus of elasticity. [2]

43. [9231/w18/22/q4]



A uniform rod AB of length $2a$ and weight W rests against a smooth horizontal peg at a point C on the rod, where $AC = x$. The lower end A of the rod rests on rough horizontal ground. The rod is in equilibrium inclined at an angle of 45° to the horizontal (see diagram). The coefficient of friction between the rod and the ground is μ. The rod is about to slip at A.

(i) Find an expression for x in terms of a and μ. [5]

(ii) Hence show that $\mu \geqslant \frac{1}{3}$. [2]

(iii) Given that $x = \frac{3}{2}a$, find the value of μ and the magnitude of the resultant force on the rod at A. [4]

44. [9709/m18/52/q1]

A uniform rectangular block has a square base $ABCD$ with $AB = BC = 0.4$ m. The height of the block is h m. The block is placed with its base on a rough plane inclined at 30° to the horizontal. The block does not slide. It is given that the block is on the point of toppling when the diagonal AC lies along a line of greatest slope. Calculate h. [3]

45. [9709/m18/52/q7]

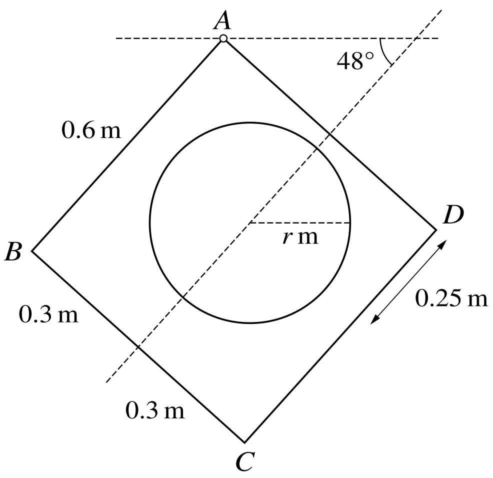


Fig. 1

ABCD is a uniform square lamina with sides of length 0.6 m. A circular hole of radius *r* m is made in the lamina. The centre of the hole is 0.3 m from *AB* and 0.25 m from *AD*. The lamina is freely suspended at *A* and hangs with the axis of symmetry making an angle of 48° with the horizontal (see Fig. 1).

(i) Show that $r = 0.214$, correct to 3 significant figures. [5]

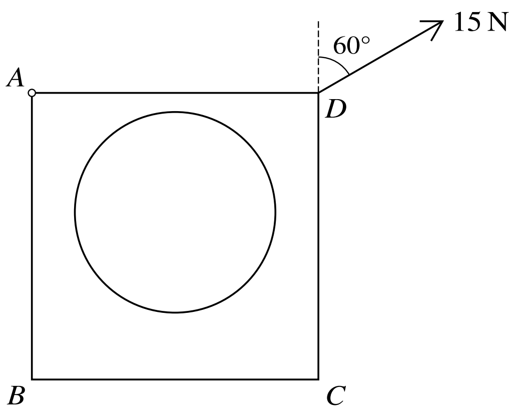
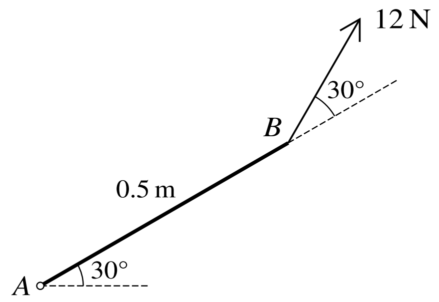


Fig. 2

The lamina is held in equilibrium with *AD* horizontal by a force of magnitude 15 N acting in the plane of the lamina applied at *D*. The line of action of this force makes an angle of 60° with the vertical (see Fig. 2).

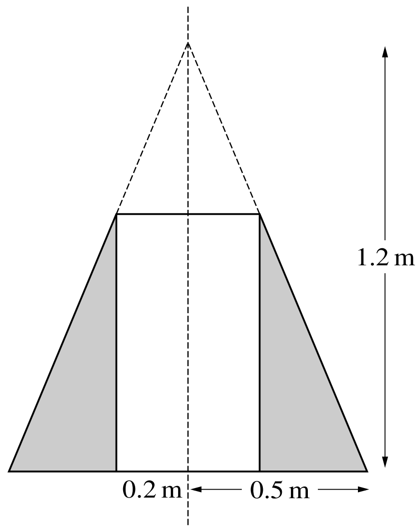
(ii) Find the weight of the original square lamina, before the hole was made. [4]

46. [9709/s18/51/q2]



A non-uniform rod AB of length 0.5 m and weight 8 N is freely hinged to a fixed point at A. The rod makes an angle of 30° with the horizontal with B above the level of A. The rod is held in equilibrium by a force of magnitude 12 N acting in the vertical plane containing the rod at an angle of 30° to AB applied at B (see diagram). Find the distance of the centre of mass of the rod from A. [3]

47. [9709/s18/51/q7]



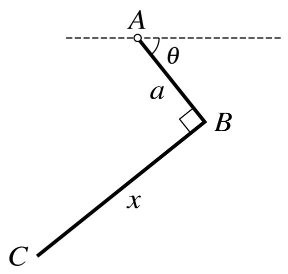
A uniform solid cone has height 1.2 m and base radius 0.5 m. A uniform object is made by drilling a cylindrical hole of radius 0.2 m through the cone along the axis of symmetry (see diagram).

(i) Show that the height of the object is 0.72 m and that the volume of the cone removed by the drilling is $0.0352\pi\,\text{m}^3$. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

(ii) Find the distance of the centre of mass of the object from its base. [6]

48. [9709/s18/52/q3]

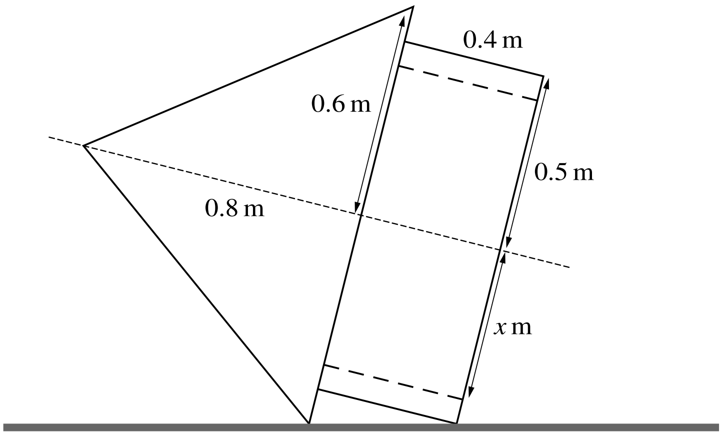


ABC is an object made from a uniform wire consisting of two straight portions *AB* and *BC*, in which $AB = a$, $BC = x$ and angle $ABC = 90°$. When the object is freely suspended from *A* and in equilibrium, the angle between *AB* and the horizontal is θ (see diagram).

(i) Show that $x^2 \tan\theta - 2ax - a^2 = 0$. [3]

(ii) Given that $\tan\theta = 1.25$, calculate the length of the wire in terms of a. [2]

49. [9709/s18/52/q5]



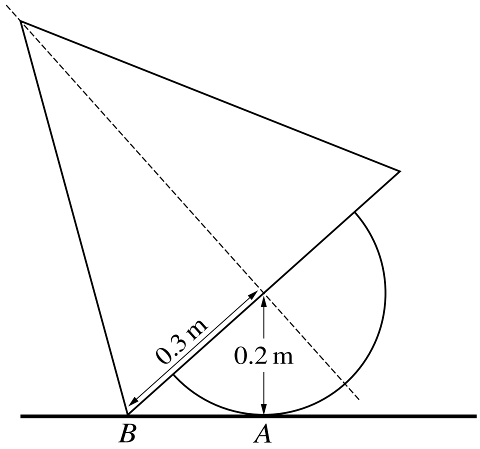
A uniform object is made by joining a solid cone of height 0.8 m and base radius 0.6 m and a cylinder. The cylinder has length 0.4 m and radius 0.5 m. The cylinder has a cylindrical hole of length 0.4 m and radius x m drilled through it along the axis of symmetry. A plane face of the cylinder is attached to the base of the cone so that the object has an axis of symmetry perpendicular to its base and passing through the vertex of the cone. The object is placed with points on the base of the cone and the base of the cylinder in contact with a horizontal surface (see diagram). The object is on the point of toppling.

(i) Show that the centre of mass of the object is 0.15 m from the base of the cone. [3]

(ii) Find x. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

50. [9709/w18/51/q2]



A uniform object is made by attaching the base of a solid hemisphere to the base of a solid cone so that the object has an axis of symmetry. The base of the cone has radius 0.3 m, and the hemisphere has radius 0.2 m. The object is placed on a horizontal plane with a point A on the curved surface of the hemisphere and a point B on the circumference of the cone in contact with the plane (see diagram).

(i) Given that the object is on the point of toppling about B, find the distance of the centre of mass of the object from the base of the cone. [3]

(ii) Given instead that the object is on the point of toppling about A, calculate the height of the cone. [3]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$. The volume of a hemisphere is $\frac{2}{3}\pi r^3$.]

51. [9709/w18/51/q6]

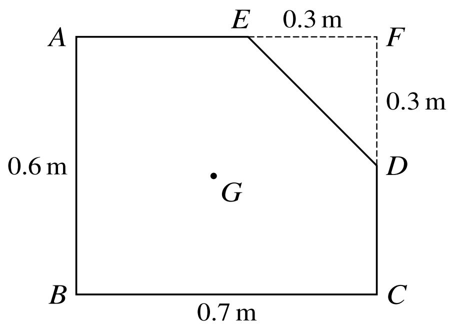


Fig. 1

Fig. 1 shows the cross-section *ABCDE* through the centre of mass *G* of a uniform prism. The cross-section consists of a rectangle *ABCF* from which a triangle *DEF* has been removed; $AB = 0.6$ m, $BC = 0.7$ m and $DF = EF = 0.3$ m.

(i) Show that the distance of *G* from *BC* is 0.276 m, and find the distance of *G* from *AB*. [5]

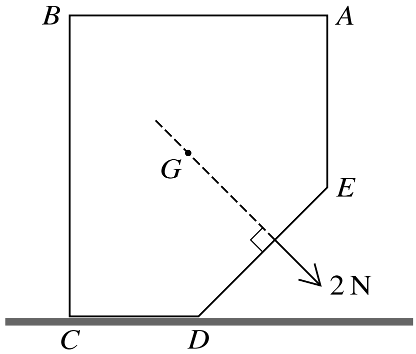


Fig. 2

The prism is placed with *CD* on a rough horizontal surface. A force of magnitude 2 N acting in the plane of the cross-section is applied to the prism. The line of action of the force passes through *G* and is perpendicular to *DE* (see Fig. 2). The prism is on the point of toppling about the edge through *D*.

(ii) Calculate the weight of the prism. [3]

52. [9709/w18/52/q2]

A uniform solid object is made by attaching a cone to a cylinder so that the circumferences of the base of the cone and a plane face of the cylinder coincide. The cone and the cylinder each have radius 0.3 m and height 0.4 m.

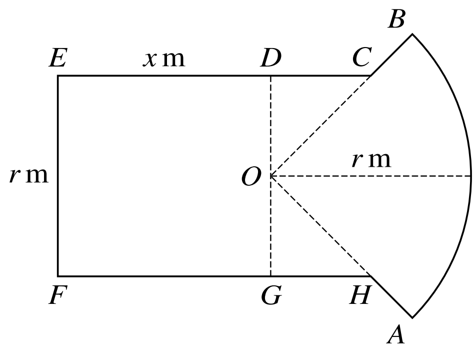
(i) Calculate the distance of the centre of mass of the object from the vertex of the cone. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

The object has weight W N and is placed with its plane circular face on a rough horizontal surface. A force of magnitude kW N acting at 30° to the upward vertical is applied to the vertex of the cone. The object does not slip.

(ii) Find the greatest possible value of k for which the object does not topple. [3]

53. [9709/w18/52/q6]

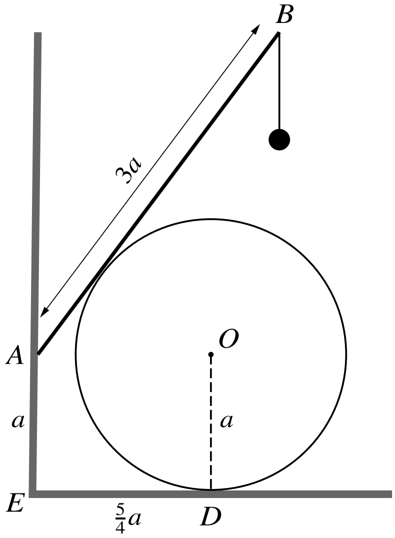


The diagram shows a uniform lamina $ABCDEFGH$. The lamina consists of a quarter-circle OAB of radius r m, a rectangle $DEFG$ and two isosceles right-angled triangles COD and GOH. The rectangle has $DG = EF = r$ m and $DE = FG = x$ m.

(i) Given that the centre of mass of the lamina is at O, express x in terms of r. [6]

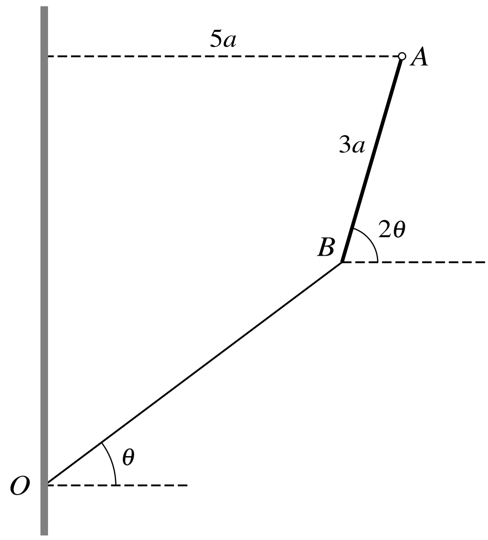
(ii) Given instead that the rectangle $DEFG$ is a square with edges of length r m, state with a reason whether the centre of mass of the lamina lies within the square or the quarter-circle. [1]

54. [9231/s17/21/q2]



A uniform smooth disc with centre O and radius a is fixed at the point D on a horizontal surface. A uniform rod of length $3a$ and weight W rests on the disc with its end A in contact with a rough vertical wall. The rod and the disc lie in a vertical plane that is perpendicular to the wall. The wall meets the horizontal surface at the point E such that $AE = a$ and $ED = \frac{5}{4}a$. A particle of weight kW is hung from the rod at B (see diagram). The coefficient of friction between the rod and the wall is $\frac{1}{8}$ and the system is in limiting equilibrium. Find the value of k. [8]

55. [9231/s17/23/q4]



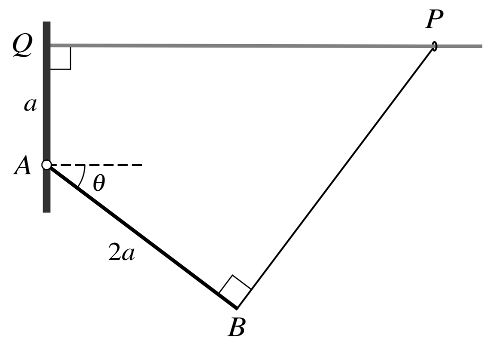
A uniform rod AB of length $3a$ and weight W is freely hinged to a fixed point at the end A. The end B is below the level of A and is attached to one end of a light elastic string of natural length $4a$. The other end of the string is attached to a point O on a vertical wall. The horizontal distance between A and the wall is $5a$. The string and the rod make angles θ and 2θ respectively with the horizontal (see diagram). The system is in equilibrium with the rod and the string in the same vertical plane. It is given that $\sin\theta = \frac{3}{5}$ and you may use the fact that $\cos 2\theta = \frac{7}{25}$.

(i) Find the tension in the string in terms of W. [3]

(ii) Find the modulus of elasticity of the string in terms of W. [4]

(iii) Find the angle that the force acting on the rod at A makes with the horizontal. [3]

56. [9231/w17/21/q4]



A small ring P of weight W is free to slide on a rough horizontal wire, one end of which is attached to a vertical wall at Q. The end A of a thin uniform rod AB of length $2a$ and weight $\frac{5}{2}W$ is freely hinged to the wall at the point A which is a distance a vertically below Q. A light elastic string of natural length $2a$ has one end attached to the ring P and the other end attached to the rod at B. The string is at right angles to the rod and A, B, P and Q lie in a vertical plane. The system is in limiting equilibrium with AB making an angle θ with the horizontal, where $\sin\theta = \frac{3}{5}$ (see diagram).

(i) Find the tension in the string in terms of W. [2]

(ii) Find the coefficient of friction between the ring and the wire. [2]

(iii) Find the magnitude of the resultant force on the rod at the hinge in terms of W. [3]

(iv) Find the modulus of elasticity of the string in terms of W. [3]

57. [9709/m17/52/q2]

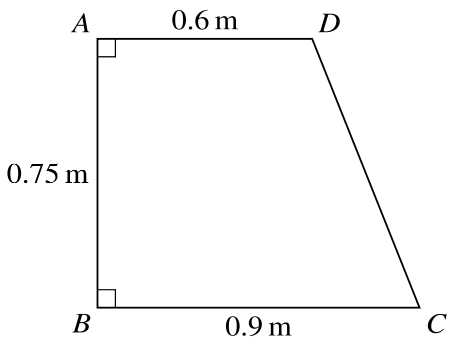
A cylindrical container is open at the top. The curved surface and the circular base of the container are both made from the same thin uniform material. The container has radius 0.2 m and height 0.9 m.

(i) Show that the centre of mass of the container is 0.405 m from the base. [3]

The container is placed with its base on a rough inclined plane. The container is in equilibrium on the point of slipping down the plane and also on the point of toppling.

(ii) Find the coefficient of friction between the container and the plane. [3]

58. [9709/m17/52/q4]



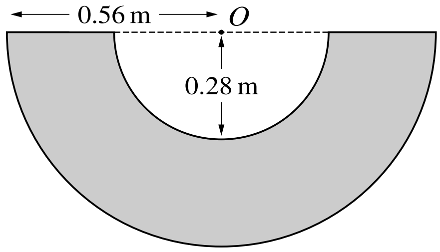
The diagram shows a uniform lamina $ABCD$ with $AB = 0.75$ m, $AD = 0.6$ m and $BC = 0.9$ m. Angle BAD = angle $ABC = 90°$.

(i) Show that the distance of the centre of mass of the lamina from AB is 0.38 m, and find the distance of the centre of mass from BC. [5]

The lamina is freely suspended at B and hangs in equilibrium.

(ii) Find the angle between BC and the vertical. [2]

59. [9709/s17/51/q3]



An object is made from a uniform solid hemisphere of radius 0.56 m and centre O by removing a hemisphere of radius 0.28 m and centre O. The diagram shows a cross-section through O of the object.

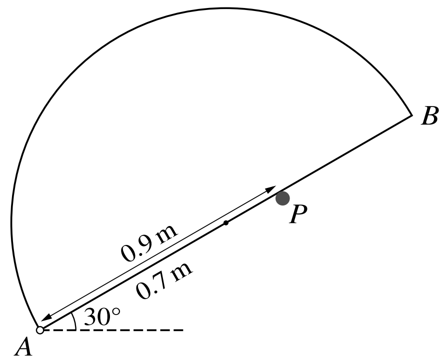
(i) Calculate the distance of the centre of mass of the object from O. [4]

[The volume of a hemisphere is $\frac{2}{3}\pi r^3$.]

The object has weight 24 N. A uniform hemisphere H of radius 0.28 m is placed in the hollow part of the object to create a non-uniform hemisphere with centre O. The centre of mass of the non-uniform hemisphere is 0.15 m from O.

(ii) Calculate the weight of H. [3]

60. [9709/s17/51/q5]



A uniform semicircular lamina of radius 0.7 m and weight 14 N has diameter AB. The lamina is in a vertical plane with A freely pivoted at a fixed point. The straight edge AB rests against a small smooth peg P above the level of A. The angle between AB and the horizontal is 30° and $AP = 0.9$ m (see diagram).

(i) Show that the magnitude of the force exerted by the peg on the lamina is 7.12 N, correct to 3 significant figures. [4]

(ii) Find the angle with the horizontal of the force exerted by the pivot on the lamina at A. [3]

61. [9709/s17/52/q3]

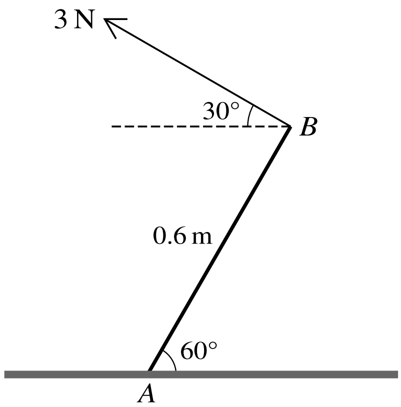
An open box in the shape of a cube with edges of length 0.2 m is placed with its base horizontal and its four sides vertical. The four sides and base are uniform laminas, each with weight 3 N.

(i) Calculate the height of the centre of mass of the box above its base. [3]

The box is now fitted with a thin uniform square lid of weight 3 N and with edges of length 0.2 m. The lid is attached to the box by a hinge of length 0.2 m and weight 2 N. The lid of the box is held partly open.

(ii) Find the angle which the lid makes with the horizontal when the centre of mass of the box (including the lid and hinge) is 0.12 m above the base of the box. [4]

62. [9709/s17/52/q6]



The end A of a non-uniform rod AB of length 0.6 m and weight 8 N rests on a rough horizontal plane, with AB inclined at 60° to the horizontal. Equilibrium is maintained by a force of magnitude 3 N applied to the rod at B. This force acts at 30° above the horizontal in the vertical plane containing the rod (see diagram).

(i) Find the distance of the centre of mass of the rod from A. [2]

The 3 N force is removed, and the rod is held in equilibrium by a force of magnitude P N applied at B, acting in the vertical plane containing the rod, at an angle of 30° below the horizontal.

(ii) Calculate P. [2]

In one of the two situations described, the rod AB is in limiting equilibrium.

(iii) Find the coefficient of friction at A. [4]

63. [9709/w17/51/q6]

A solid object consists of a uniform hemisphere of radius 0.4 m attached to a uniform cylinder of radius 0.4 m so that the circumferences of their circular faces coincide. The hemisphere and cylinder each have weight 20 N. The centre of mass of the object lies at the centre O of their common circular face.

(i) Show that the height of the cylinder is 0.3 m. [2]

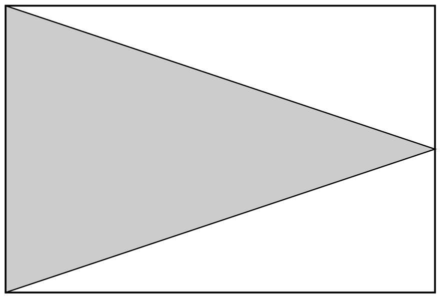
A new object is made by cutting the cylinder in half and removing the half not attached to the hemisphere. The cut is perpendicular to the axis of symmetry, so the new object consists of a hemisphere and a cylinder half the height of the original cylinder.

(ii) Find the distance of the centre of mass of the new object from O. [4]

The new object is placed with its hemispherical part on a rough horizontal surface. The new object is held in equilibrium by a force of magnitude P N acting along its axis of symmetry, which is inclined at 30° to the horizontal.

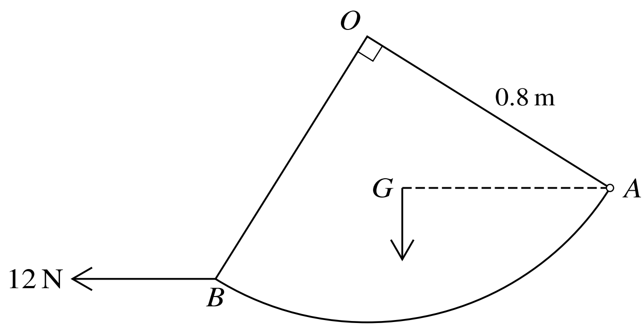
(iii) Find P. [3]

64. [9709/w17/52/q2]



A uniform solid cone has height 0.6 m and base radius 0.2 m. A uniform hollow cylinder, open at both ends, has the same dimensions. An object is made by putting the cone inside the cylinder so that the base of the cone coincides with one end of the cylinder (see diagram, which shows a cross-section). The total weight of the object is 60 N and its centre of mass is 0.25 m from the base of the cone. Calculate the weight of the cone. [3]

65. [9709/w17/52/q5]

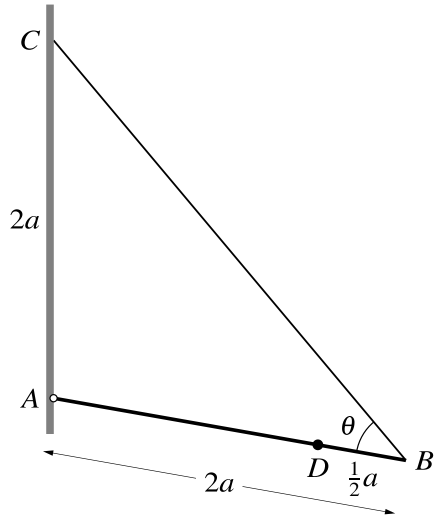


OAB is a uniform lamina in the shape of a quadrant of a circle with centre *O* and radius 0.8 m which has its centre of mass at *G*. The lamina is smoothly hinged at *A* to a fixed point and is free to rotate in a vertical plane. A horizontal force of magnitude 12 N acting in the plane of the lamina is applied to the lamina at *B*. The lamina is in equilibrium with *AG* horizontal (see diagram).

(i) Calculate the length *AG*. [3]

(ii) Find the weight of the lamina. [5]

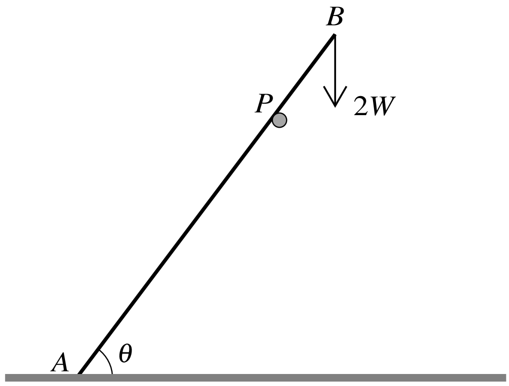
66. [9231/s16/21/q11e]



The end A of a uniform rod AB, of length $2a$ and weight W, is freely hinged to a vertical wall. The end B of the rod is attached to a light elastic string of natural length $\frac{3}{2}a$ and modulus of elasticity $3W$. The other end of the string is attached to the point C on the wall, where C is vertically above A and $AC = 2a$. A particle of weight $2W$ is attached to the rod at the point D, where $DB = \frac{1}{2}a$. The angle ABC is equal to θ (see diagram). Show that $\cos\theta = \frac{3}{4}$ and find the tension in the string in terms of W. [10]

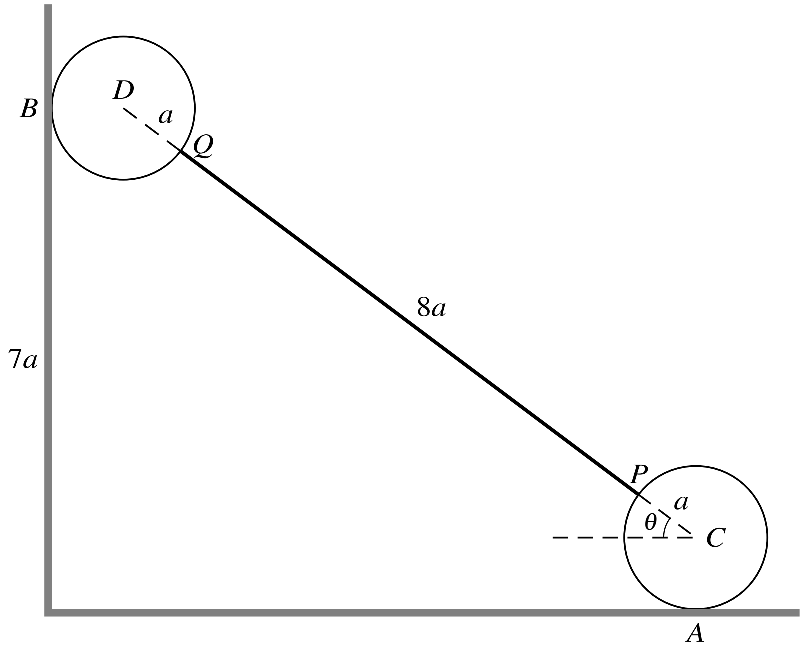
Find the magnitude of the reaction force at the hinge. [4]

67. [9231/s16/23/q2]



A uniform rod AB of length $2a$ and weight W rests with its lower end A in contact with a rough horizontal surface. The rod rests on a smooth peg at the point P. A particle of weight $2W$ is suspended from the end B of the rod. The system is in limiting equilibrium with the angle between the rod and the horizontal surface equal to θ, where $\tan\theta = \frac{4}{3}$ (see diagram). The coefficient of friction between the rod and the surface is $\frac{2}{3}$. Find the distance AP. [8]

68. [9231/w16/21/q3]



The end P of a uniform rod PQ, of weight kW and length $8a$, is rigidly attached to a point on the surface of a uniform sphere with centre C, weight W and radius a. The end Q is rigidly attached to a point on the surface of an identical sphere with centre D. The points C, P, Q and D are in a straight line. The object consisting of the rod and two spheres rests with one sphere in contact with a rough horizontal surface, at the point A, and the other sphere in contact with a smooth vertical wall, at the point B. The angle between CD and the horizontal is θ. The point B is at a height of $7a$ above the base of the wall (see diagram). The points A, B, C, D, P and Q are all in the same vertical plane.

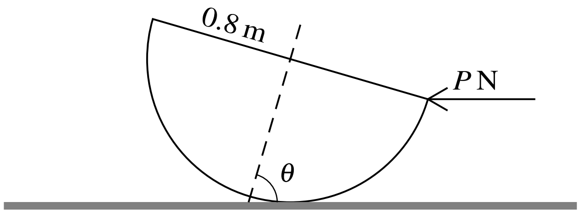
(i) Show that $\sin\theta = \frac{3}{5}$. [1]

The object is in limiting equilibrium and the coefficient of friction at A is μ.

(ii) Find the numerical value of μ. [7]

(iii) Given that the resultant force on the object at A is $W\sqrt{(65)}$, show that $k = 5$. [3]

69. [9709/m16/52/q2]

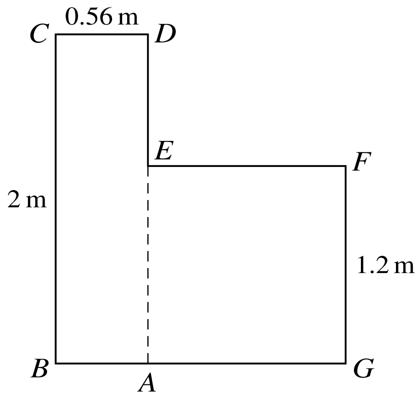


A uniform solid hemisphere of weight 60 N and radius 0.8 m rests in limiting equilibrium with its curved surface on a rough horizontal plane. The axis of symmetry of the hemisphere is inclined at an angle of θ to the horizontal, where $\cos\theta = 0.28$. Equilibrium is maintained by a horizontal force of magnitude P N applied to the lowest point of the circular rim of the hemisphere (see diagram).

(i) Show that $P = 8.75$. [3]

(ii) Find the coefficient of friction between the hemisphere and the plane. [2]

70. [9709/m16/52/q4]



A uniform lamina is made by joining a rectangle $ABCD$, in which $AB = CD = 0.56$ m and $BC = AD = 2$ m, and a square $EFGA$ of side 1.2 m. The vertex E of the square lies on the edge AD of the rectangle (see diagram). The centre of mass of the lamina is a distance h m from BC and a distance v m from BAG.

(i) Find the value of h and show that $v = h$. [4]

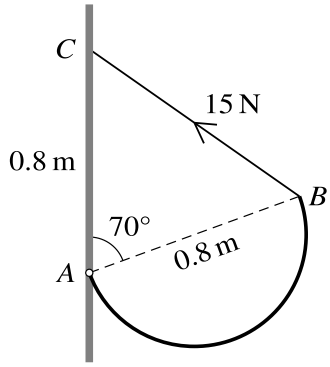
The lamina is freely suspended at the point B and hangs in equilibrium.

(ii) State the angle which the edge BC makes with the horizontal. [1]

Instead, the lamina is now freely suspended at the point F and hangs in equilibrium.

(iii) Calculate the angle between FG and the vertical. [2]

71. [9709/s16/51/q2]

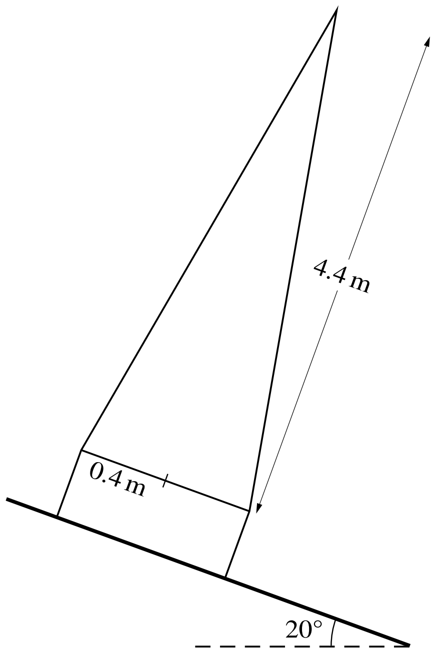


A uniform wire has the shape of a semicircular arc, with diameter AB of length 0.8 m. The wire is attached to a vertical wall by a smooth hinge at A. The wire is held in equilibrium with AB inclined at 70° to the upward vertical by a light string attached to B. The other end of the string is attached to the point C on the wall 0.8 m vertically above A. The tension in the string is 15 N (see diagram).

(i) Show that the horizontal distance of the centre of mass of the wire from the wall is 0.463 m, correct to 3 significant figures. [3]

(ii) Calculate the weight of the wire. [2]

72. [9709/s16/51/q4]

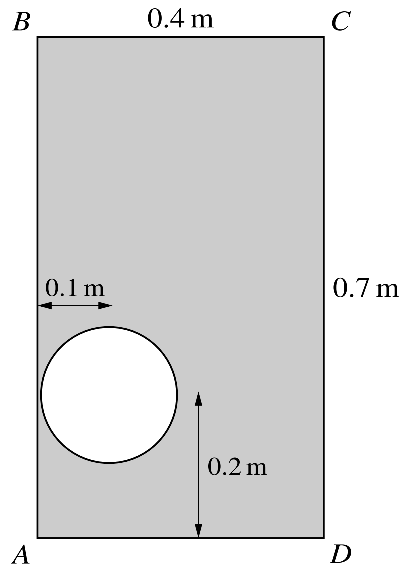


A uniform solid cone has base radius 0.4 m and height 4.4 m. A uniform solid cylinder has radius 0.4 m and weight equal to the weight of the cone. An object is formed by attaching the cylinder to the cone so that the base of the cone and a circular face of the cylinder are in contact and their circumferences coincide. The object rests in equilibrium with its circular base on a plane inclined at an angle of 20° to the horizontal (see diagram).

(i) Calculate the least possible value of the coefficient of friction between the plane and the object. [2]

(ii) Calculate the greatest possible height of the cylinder. [4]

73. [9709/s16/52/q4]



A uniform object is made by drilling a cylindrical hole through a rectangular block. The axis of the cylindrical hole is perpendicular to the cross-section $ABCD$ through the centre of mass of the object. $AB = CD = 0.7$ m, $BC = AD = 0.4$ m, and the centre of the hole is 0.1 m from AB and 0.2 m from AD (see diagram). The hole has a cross-section of area 0.03 m^2.

(i) Show that the distance of the centre of mass of the object from AB is 0.212 m, and calculate the distance of the centre of mass from AD. [4]

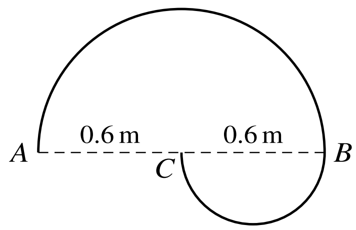
The object has weight 70 N and is placed on a rough horizontal surface, with AD in contact with the surface. A vertically upwards force of magnitude F N acts on the object at C. The object is on the point of toppling.

(ii) Find the value of F. [2]

The force acting at C is removed, and the object is placed on a rough plane inclined at an angle $\theta°$ to the horizontal. AD lies along a line of greatest slope, with A higher than D. The plane is sufficiently rough to prevent sliding, and the object does not topple.

(iii) Find the greatest possible value of θ. [2]

74. [9709/w16/51/q2]



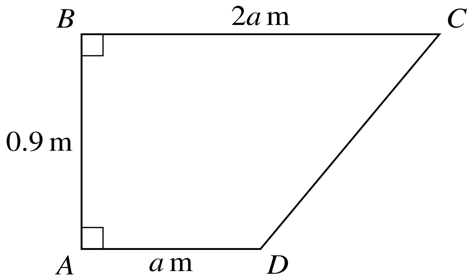
A uniform wire is bent to form an object which has a semicircular arc with diameter AB of length 1.2 m, with a smaller semicircular arc with diameter BC of length 0.6 m. The end C of the smaller arc is at the centre of the larger arc (see diagram). The two semicircular arcs of the wire are in the same plane.

(i) Show that the distance of the centre of mass of the object from the line ACB is 0.191 m, correct to 3 significant figures. [3]

The object is freely suspended at A and hangs in equilibrium.

(ii) Find the angle between ACB and the vertical. [4]

75. [9709/w16/51/q4]



The diagram shows the cross-section $ABCD$ through the centre of mass of a uniform solid prism. $AB = 0.9$ m, $BC = 2a$ m, $AD = a$ m and angle ABC = angle $BAD = 90°$.

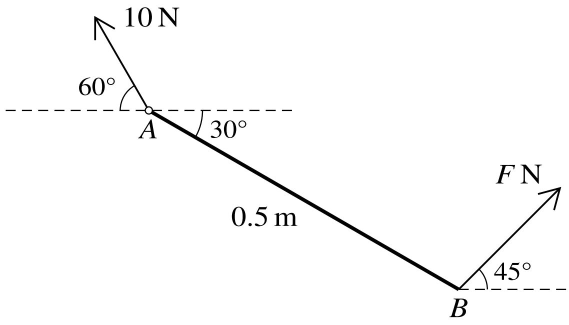
(i) Calculate the distance of the centre of mass of the prism from AD. [2]

(ii) Express the distance of the centre of mass of the prism from AB in terms of a. [2]

The prism has weight 18 N and rests in equilibrium on a rough horizontal surface, with AD in contact with the surface. A horizontal force of magnitude 6 N is applied to the prism. This force acts through the centre of mass in the direction BC.

(iii) Given that the prism is on the point of toppling, calculate a. [3]

76. [9709/w16/52/q3]

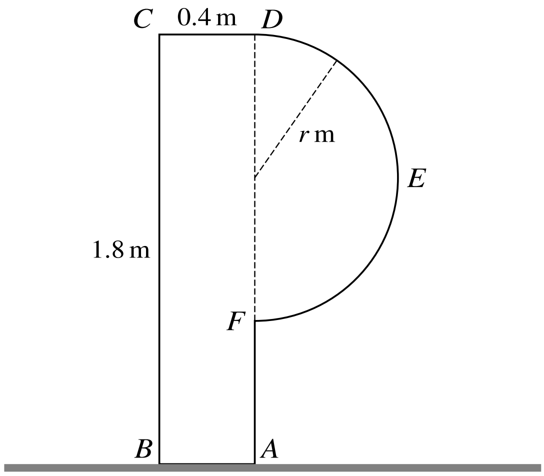


A non-uniform rod *AB* of length 0.5 m is freely hinged to a fixed point at *A*. The rod is in equilibrium at an angle of 30° with the horizontal with *B* below the level of *A*. Equilibrium is maintained by a force of magnitude *F* N applied at *B* acting at 45° above the horizontal in the vertical plane containing *AB*. The force exerted by the hinge on the rod has magnitude 10 N and acts at an angle of 60° above the horizontal (see diagram).

(i) By resolving horizontally and vertically, calculate *F* and the weight of the rod. [4]

(ii) Find the distance of the centre of mass of the rod from *A*. [3]

77. [9709/w16/52/q6]



The diagram shows the cross-section $ABCDEF$ through the centre of mass of a uniform prism which rests with AB on rough horizontal ground. $ABCD$ is a rectangle with $AB = CD = 0.4$ m and $BC = AD = 1.8$ m. The other part of the cross-section is a semicircle with diameter DF and radius r m.

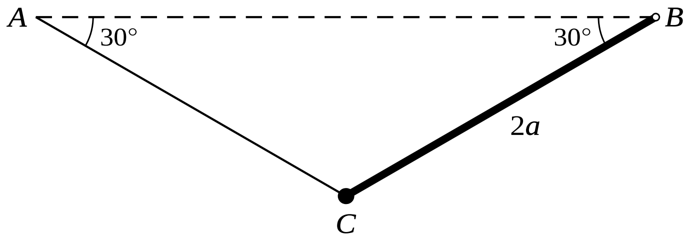
(i) Given that the prism is on the point of toppling, show that $r = 0.6$. [3]

A force of magnitude P N is applied to the prism, acting at 60° to the upwards vertical along a tangent to the semicircle at a point between D and E. The prism has weight 15 N and is in equilibrium on the point of toppling about B.

(ii) Show that $P = 3.26$, correct to 3 significant figures. [4]

(iii) Find the smallest possible value of the coefficient of friction between the prism and the ground. [2]

78. [9231/s15/21/q4]

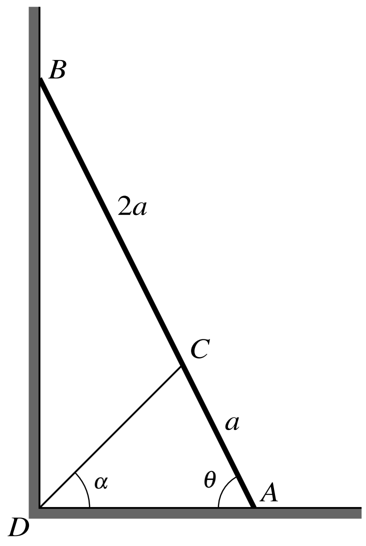


A uniform rod BC of length $2a$ and weight W is hinged to a fixed point at B. A particle of weight $3W$ is attached to the rod at C. The system is held in equilibrium by a light elastic string of natural length $\frac{3}{5}a$ in the same vertical plane as the rod. One end of the elastic string is attached to the rod at C and the other end is attached to a fixed point A which is at the same horizontal level as B. The rod and the string each make an angle of 30° with the horizontal (see diagram). Find

(i) the modulus of elasticity of the string, [5]

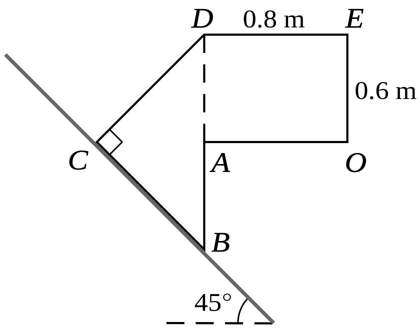
(ii) the magnitude and direction of the force acting on the rod at B. [5]

79. [9231/w15/21/q1]



A uniform ladder AB, of length $3a$ and weight W, rests with the end A in contact with smooth horizontal ground and the end B against a smooth vertical wall. One end of a light inextensible rope is attached to the ladder at the point C, where $AC = a$. The other end of the rope is fixed to the point D at the base of the wall and the rope DC is in the same vertical plane as the ladder AB. The ladder rests in equilibrium in a vertical plane perpendicular to the wall, with the ladder making an angle θ with the horizontal and the rope making an angle α with the horizontal (see diagram). It is given that $\tan\theta = 2\tan\alpha$. Find, in terms of W and α, the tension in the rope and the magnitudes of the forces acting on the ladder at A and at B. [9]

80. [9709/s15/51/q7]



The diagram shows the cross-section *OABCDE* through the centre of mass of a uniform prism on a rough inclined plane. The portion *ADEO* is a rectangle in which $AD = OE = 0.6$ m and $DE = AO = 0.8$ m; the portion *BCD* is an isosceles triangle in which angle *BCD* is a right angle, and *A* is the mid-point of *BD*. The plane is inclined at 45° to the horizontal, *BC* lies along a line of greatest slope of the plane and *DE* is horizontal.

(i) Calculate the distance of the centre of mass of the prism from *BD*. [3]

The weight of the prism is 21 N, and it is held in equilibrium by a horizontal force of magnitude *P* N acting along *ED*.

(ii) **(a)** Find the smallest value of *P* for which the prism does not topple. [2]

(b) It is given that the prism is about to slip for this smallest value of *P*. Calculate the coefficient of friction between the prism and the plane. [3]

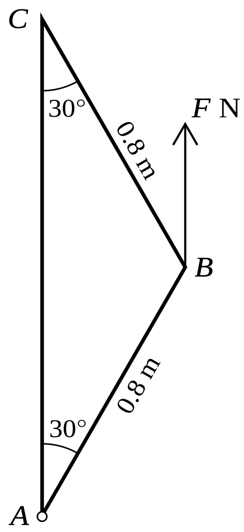
The value of *P* is gradually increased until the prism ceases to be in equilibrium.

(iii) Show that the prism topples before it begins to slide, stating the value of *P* at which equilibrium is broken. [5]

81. [9709/s15/52/q3]

A triangular frame ABC consists of two uniform rigid rods each of length 0.8 m and weight 3 N, and a longer uniform rod of weight 4 N. The triangular frame has $AB = BC$, and angle BAC = angle BCA = 30°.

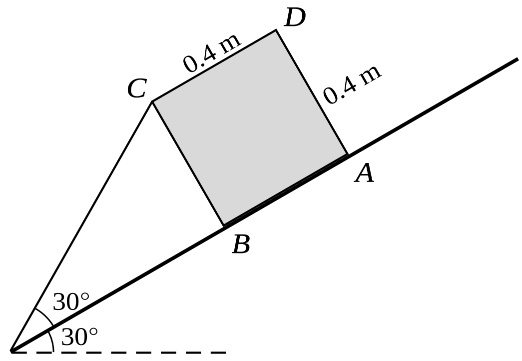
(i) Calculate the distance of the centre of mass of the frame from AC. [3]



The vertex A of the frame is attached to a smooth hinge at a fixed point. The frame is held in equilibrium with AC vertical by a vertical force of magnitude F N applied to the frame at B (see diagram).

(ii) Calculate F, and state the magnitude and direction of the force acting on the frame at the hinge. [3]

82. [9709/s15/52/q5]



A uniform solid cube with edges of length 0.4 m rests in equilibrium on a rough plane inclined at an angle of 30° to the horizontal. *ABCD* is a cross-section through the centre of mass of the cube, with *AB* along a line of greatest slope. *B* lies below the level of *A*. One end of a light elastic string with modulus of elasticity 12 N and natural length 0.4 m is attached to *C*. The other end of the string is attached to a point below the level of *B* on the same line of greatest slope, such that the string makes an angle of 30° with the plane (see diagram). The cube is on the point of toppling. Find

(i) the tension in the string, [3]

(ii) the weight of the cube. [4]

83. [9709/s15/53/q1]

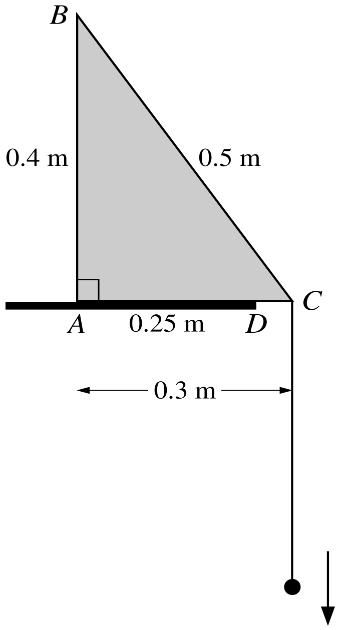
A uniform semicircular lamina has diameter AB of length 0.8 m.

(i) Find the distance of the centre of mass of the lamina from AB. [2]

The lamina rests in a vertical plane, with the point B of the lamina in contact with a rough horizontal surface and with A vertically above B. Equilibrium is maintained by a force of magnitude 6 N in the plane of the lamina, applied to the lamina at A and acting at an angle of 20° below the horizontal.

(ii) Calculate the mass of the lamina. [3]

84. [9709/s15/53/q5]

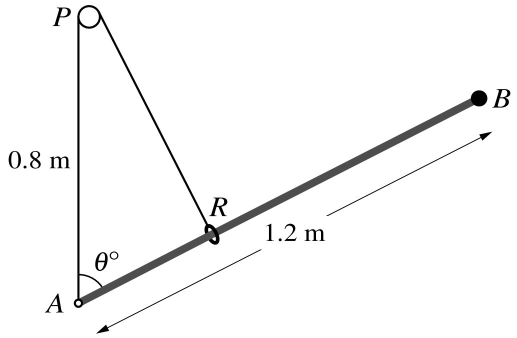


A uniform triangular prism of weight 20 N rests on a horizontal table. ABC is the cross-section through the centre of mass of the prism, where $BC = 0.5$ m, $AB = 0.4$ m, $AC = 0.3$ m and angle $BAC = 90°$. The vertical plane ABC is perpendicular to the edge of the table. The point D on AC is at the edge of the table, and $AD = 0.25$ m. One end of a light elastic string of natural length 0.6 m and modulus of elasticity 48 N is attached to C and a particle of mass 2.5 kg is attached to the other end of the string. The particle is released from rest at C and falls vertically (see diagram).

(i) Show that the tension in the string is 60 N at the instant when the prism topples. [3]

(ii) Calculate the speed of the particle at the instant when the prism topples. [5]

85. [9709/w15/51/q2]



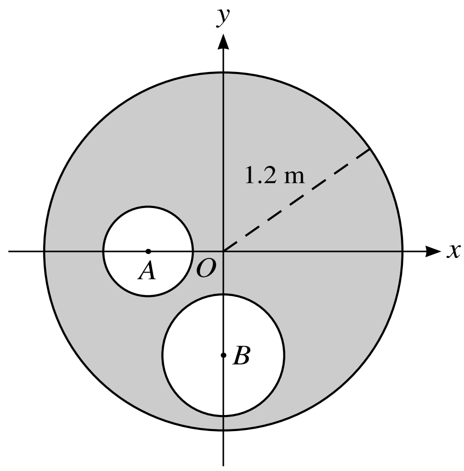
A uniform rigid rod AB of length 1.2 m and weight 8 N has a particle of weight 2 N attached at the end B. The end A of the rod is freely hinged to a fixed point. One end of a light elastic string of natural length 0.8 m and modulus of elasticity 20 N is attached to the hinge. The string passes over a small smooth pulley P fixed 0.8 m vertically above the hinge. The other end of the string is attached to a small light smooth ring R which can slide on the rod. The system is in equilibrium with the rod inclined at an angle $\theta°$ to the vertical (see diagram).

(i) Show that the tension in the string is $20 \sin \theta$ N. [1]

(ii) Explain why the part of the string attached to the ring is perpendicular to the rod. [1]

(iii) Find θ. [3]

86. [9709/w15/51/q6]



A uniform circular disc has centre O and radius 1.2 m. The centre of the disc is at the origin of x- and y-axes. Two circular holes with centres at A and B are made in the disc (see diagram). The point A is on the negative x-axis with $OA = 0.5$ m. The point B is on the negative y-axis with $OB = 0.7$ m. The hole with centre A has radius 0.3 m and the hole with centre B has radius 0.4 m. Find the distance of the centre of mass of the object from

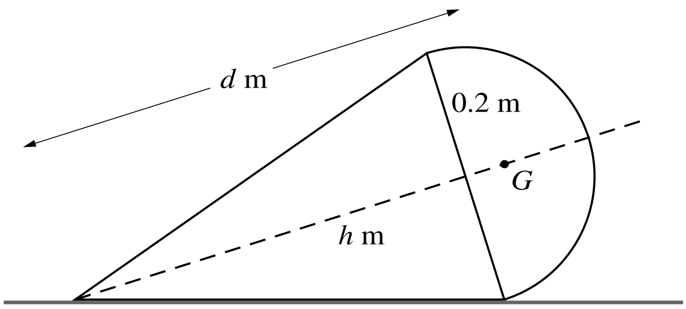
(i) the x-axis, [4]

(ii) the y-axis. [3]

The object can rotate freely in a vertical plane about a horizontal axis through O.

(iii) Calculate the angle which OA makes with the vertical when the object rests in equilibrium. [2]

87. [9709/w15/53/q6]



An object is formed by joining a hemispherical shell of radius 0.2 m and a solid cone with base radius 0.2 m and height h m along their circumferences. The centre of mass, G, of the object is d m from the vertex of the cone on the axis of symmetry of the object. The object rests in equilibrium on a horizontal plane, with the curved surface of the cone in contact with the plane (see diagram). The object is on the point of toppling.

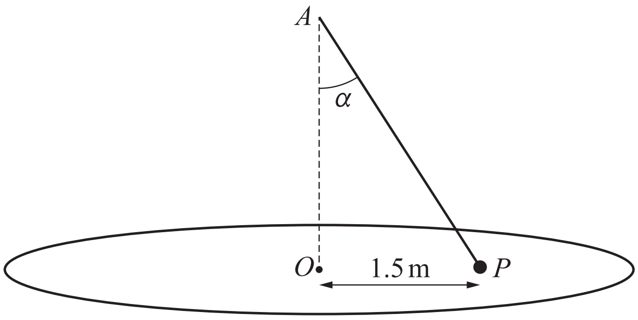
(i) Show that $d = h + \dfrac{0.04}{h}$. [3]

(ii) It is given that the cone is uniform and of weight 4 N, and that the hemispherical shell is uniform and of weight W N. Given also that $h = 0.8$, find W. [6]

Chapter 3

Circular motion (horizontal)

1. [9231/s25/31/q3]



A rough horizontal disc, centre O, rotates with constant angular speed $\omega \text{ rad s}^{-1}$. A particle P of mass 1.6 kg lies on the disc at a distance 1.5 m from O, and is attached to a point A vertically above O by a light elastic string. The string has natural length 2 m, modulus of elasticity 32 N and makes an angle α with the vertical OA (see diagram). Particle P moves in a horizontal circle also at a constant angular speed $\omega \text{ rad s}^{-1}$. Particle P is on the point of slipping in the direction OP. The coefficient of friction between the particle and the disc is 0.5.

(a) Given that the tension in the string is 8 N, show that $\sin\alpha = 0.6$. [2]

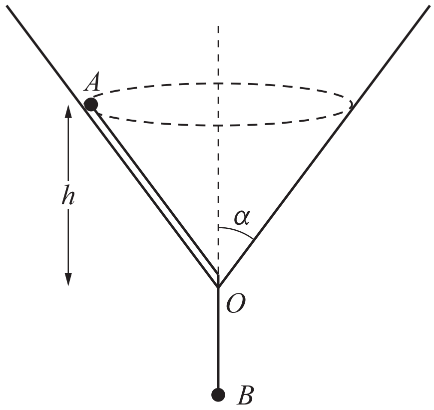
(b) Find the number of revolutions per minute made by the disc and the particle P. [5]

2. [9231/s25/33/q1]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in a horizontal circle with constant angular speed ω and with the string inclined at an angle of θ to the downward vertical.

Given that $\tan\theta = \frac{4}{3}$, find ω in terms of a and g. [3]

3. [9231/s25/34/q4]



A hollow cone with a smooth inner surface is fixed with its vertex O downwards. The semi-vertical angle of the cone is α, where $\tan\alpha = \frac{3}{4}$. A light inextensible string has a particle A of mass m attached to one end and a particle B of mass m attached to the other end. The string passes through a small hole in the cone at O. Particle B hangs in equilibrium below O. Particle A is on the inner surface of the cone at a height h above the level of O and moves in horizontal circles with constant angular speed ω (see diagram).

Find ω in terms of g and h. [6]

4. [9231/w25/31/q2]

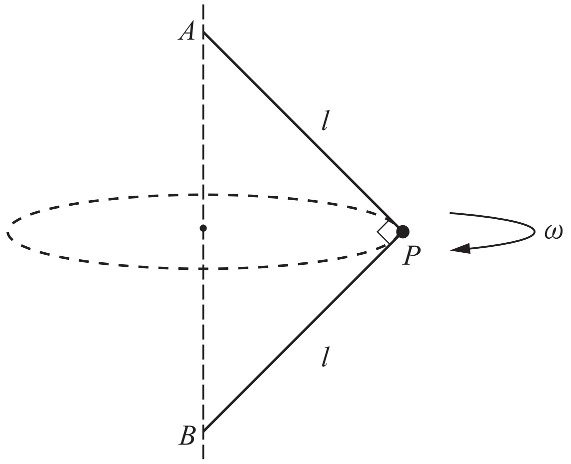
A particle P of mass m is moving in a horizontal circle with angular speed ω_1 on the smooth inner surface of a hemispherical shell of radius r. The angle between the upward vertical and the normal reaction of the surface on P is θ_1, where $\tan\theta_1 = \frac{3}{4}$.

When the angular speed is increased to ω_2, the angle between the upward vertical and the normal reaction of the surface on P becomes θ_2, where $\tan\theta_2 = \frac{4}{3}$.

Find the ratio $\dfrac{\omega_1}{\omega_2}$. [4]

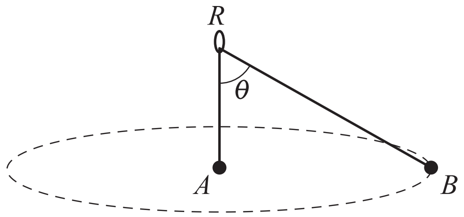
5. [9231/w25/32/q1]

A particle P of mass m is attached to two light inextensible strings each of length l. The end of one string is attached to a fixed point A and the end of the other string is attached to a fixed point B, with A vertically above B. Angle APB is a right angle. The particle P rotates in a horizontal circle at a constant angular speed ω with both strings taut (see diagram).



Find the tension in string AP in terms of m, g, l and ω. [4]

6. [9231/w25/34/q1]



A light inextensible string of length $12a$ is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass m. The other end of the string is attached to a particle B of mass $0.5m$. Particle A hangs in equilibrium vertically below the ring. Particle B moves with constant angular speed ω in a horizontal circle with particle A at its centre. The angle between AR and BR is θ (see diagram).

Express ω in terms of g and a. [5]

7. [9231/s24/31/q5]

Two particles A and B of masses m and km respectively are connected by a light inextensible string of length a. The particles are placed on a rough horizontal circular turntable with the string taut and lying along a radius of the turntable. Particle A is at a distance a from the centre of the turntable and particle B is at a distance $2a$ from the centre of the turntable. The coefficient of friction between each particle and the turntable is $\frac{1}{5}$.

When the turntable is made to rotate with angular speed $\frac{2}{5}\sqrt{\frac{g}{a}}$, the system is in limiting equilibrium.

(a) Find the tension in the string, in terms of m and g. [4]

(b) Find the value of k. [3]

8. [9231/s24/33/q2]

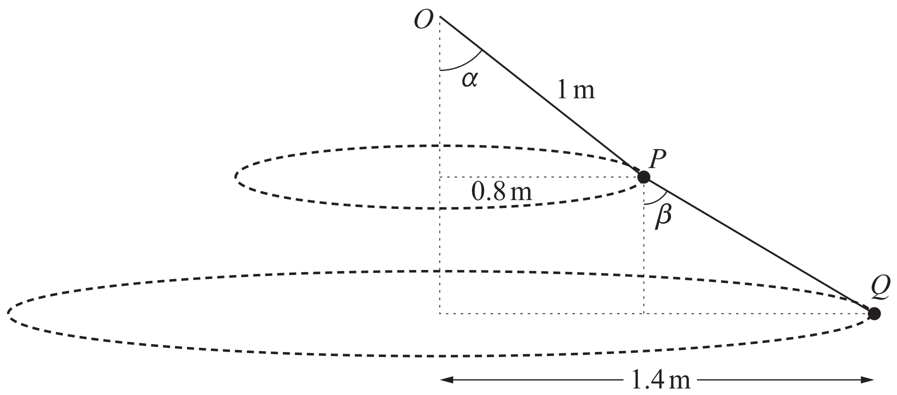
A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $2mg$. A particle Q of mass km is attached to the other end of the string. Particle P lies on a smooth horizontal table. The string has part of its length in contact with the table and then passes through a small smooth hole H in the table.

Particle P moves in a horizontal circle on the surface of the table with constant speed $\sqrt{\frac{1}{2}ga}$. Particle Q hangs in equilibrium vertically below the hole with $HQ = \frac{1}{4}a$.

(a) Find, in terms of a, the extension in the string. [4]

(b) Find the value of k. [2]

9. [9231/w24/31/q6]



A particle P of mass 0.05 kg is attached to one end of a light inextensible string of length 1 m. The other end of the string is attached to a fixed point O. A particle Q of mass 0.04 kg is attached to one end of a second light inextensible string. The other end of this string is attached to P.

The particle P moves in a horizontal circle of radius 0.8 m with angular speed ω rad s^{-1}. The particle Q moves in a horizontal circle of radius 1.4 m also with angular speed ω rad s^{-1}. The centres of the circles are vertically below O, and O, P and Q are always in the same vertical plane. The strings OP and PQ remain at constant angles α and β respectively to the vertical (see diagram).

(a) Find the tension in the string OP. [3]

(b) Find the value of ω. [3]

(c) Find the value of β. [2]

10. [9231/w24/32/q1]

A particle of mass 2 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 100 N. The other end of the string is attached to a fixed point *O* on a smooth horizontal surface. The particle is moving in a horizontal circle about *O* with the string taut and with constant angular speed 5 radians per second.

Find the extension of the string. [3]

11. [9231/s23/31/q5]

A light elastic string of natural length a and modulus of elasticity λmg has one end attached to a fixed point O on a smooth horizontal surface. When a particle of mass m is attached to the free end of the string, it moves with speed v in a horizontal circle with centre O and radius x. When, instead, a particle of mass $2m$ is attached to the free end of the string, this particle moves with speed $\frac{1}{2}v$ in a horizontal circle with centre O and radius $\frac{3}{4}x$.

(a) Find x in terms of a. [5]

(b) Given that $v = \sqrt{12ag}$, find the value of λ. [2]

12. [9231/s23/33/q5]

One end of a light elastic string, of natural length $12a$ and modulus of elasticity kmg, is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The particle moves with constant speed $\frac{3}{2}\sqrt{3ag}$ in a horizontal circle with centre at a distance $12a$ below O. The string is inclined at an angle θ to the downward vertical through O.

(a) Find, in terms of a, the extension of the string. [5]

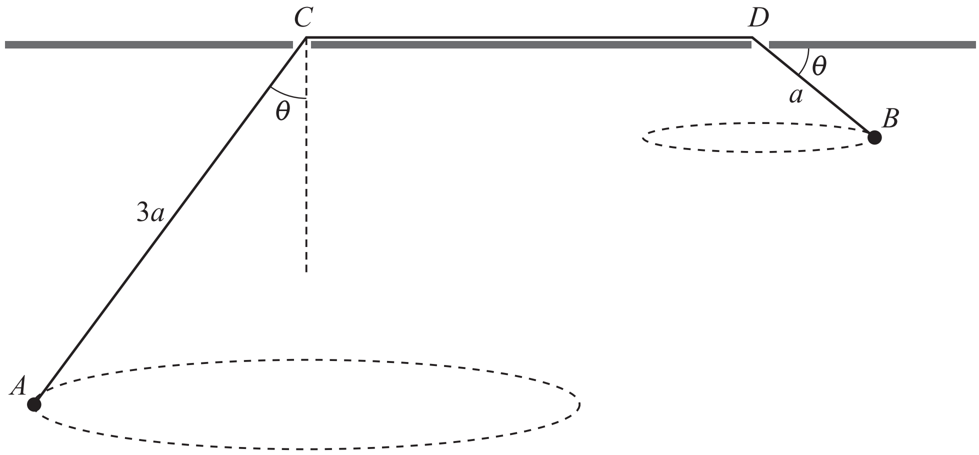
(b) Find the value of k. [3]

13. [9231/w23/32/q1]

One end of a light inextensible string of length a is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The string is taut and makes an angle θ with the downward vertical through O, where $\cos\theta = \frac{2}{3}$. The particle moves in a horizontal circle with speed v.

Find v in terms of a and g. [4]

14. [9231/s22/31/q5]



A light inextensible string AB passes through two small holes C and D in a smooth horizontal table where $AC = 3a$ and $DB = a$. A particle of mass m is attached at the end A and moves in a horizontal circle with angular velocity ω. A particle of mass $\frac{3}{4}m$ is attached to the end B and moves in a horizontal circle with angular velocity $k\omega$. AC makes an angle θ with the downward vertical and DB makes an angle θ with the horizontal (see diagram).

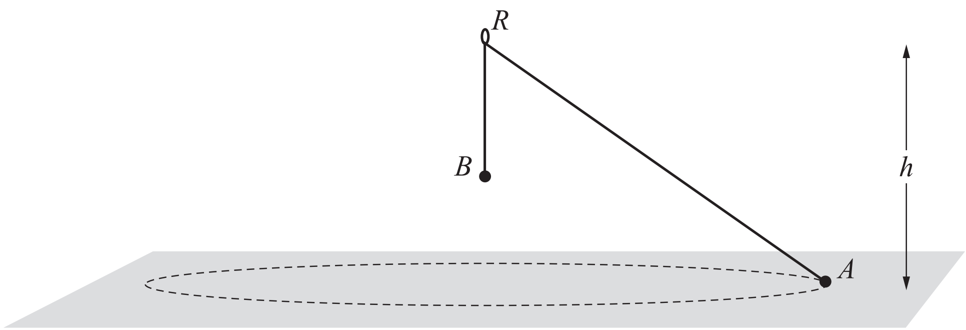
Find the value of k. [7]

15. [9231/w22/31/q1]

A particle of mass 2 kg is attached to one end of a light inextensible string of length 0.6 m. The other end of the string is attached to a fixed point on a smooth horizontal surface. The particle is moving in a circular path on the surface. The tension in the string is 20 N.

Find how many revolutions the particle makes per minute. [3]

16. [9231/w22/32/q6]



A light inextensible string is threaded through a fixed smooth ring R which is at a height h above a smooth horizontal surface. One end of the string is attached to a particle A of mass m. The other end of the string is attached to a particle B of mass $\frac{6}{7}m$. The particle A moves in a horizontal circle on the surface. The particle B hangs in equilibrium below the ring and above the surface (see diagram).

When A has constant angular speed ω, the angle between AR and BR is θ and the normal reaction between A and the surface is N.

When A has constant angular speed $\frac{3}{2}\omega$, the angle between AR and BR is α and the normal reaction between A and the surface is $\frac{1}{2}N$.

(a) Show that $\cos\theta = \frac{4}{9}\cos\alpha$. [5]

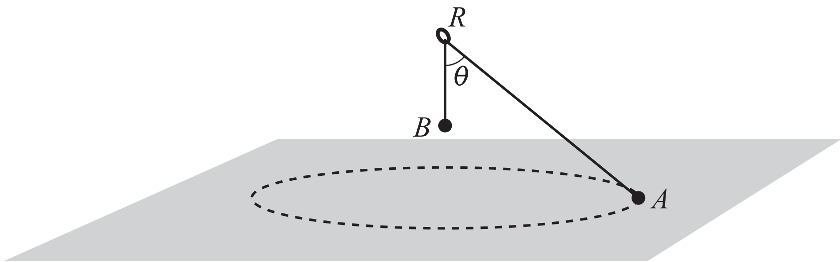
(b) Find N in terms of m and g and find the value of $\cos\alpha$. [4]

17. [9231/s21/31/q2]

A hollow hemispherical bowl of radius a has a smooth inner surface and is fixed with its axis vertical. A particle P of mass m moves in horizontal circles on the inner surface of the bowl, at a height x above the lowest point of the bowl. The speed of P is $\sqrt{\frac{8}{3}ga}$.

Find x in terms of a. [6]

18. [9231/s21/33/q3]

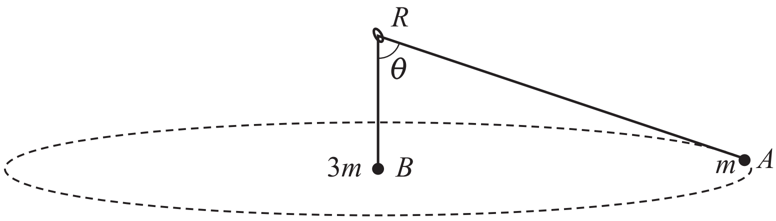


Particles A and B, of masses $3m$ and m respectively, are connected by a light inextensible string of length a that passes through a fixed smooth ring R. Particle B hangs in equilibrium vertically below the ring. Particle A moves in horizontal circles on a smooth horizontal surface with speed $\frac{2}{5}\sqrt{ga}$. The angle between AR and BR is θ (see diagram). The normal reaction between A and the surface is $\frac{12}{5}mg$.

(a) Find $\cos\theta$. [3]

(b) Find, in terms of a, the distance of B below the ring. [3]

19. [9231/w21/32/q3]

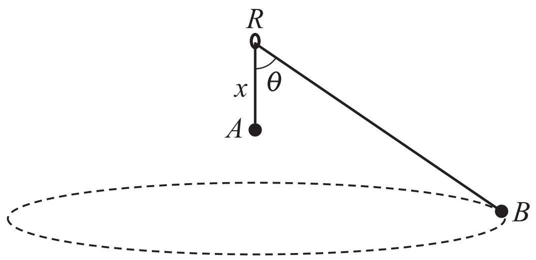


Particles A and B, of masses m and $3m$ respectively, are connected by a light inextensible string of length a that passes through a fixed smooth ring R. Particle B hangs in equilibrium vertically below the ring. Particle A moves in horizontal circles with speed v. Particles A and B are at the same horizontal level. The angle between AR and BR is θ (see diagram).

(a) Show that $\cos\theta = \frac{1}{3}$. [2]

(b) Find an expression for v in terms of a and g. [4]

20. [9231/s20/31/q2]



A light inextensible string of length a is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass $3m$. The other end of the string is attached to a particle B of mass m. The particle A hangs in equilibrium at a distance x vertically below the ring. The angle between AR and BR is θ (see diagram). The particle B moves in a horizontal circle with constant angular speed $2\sqrt{\frac{g}{a}}$.

Show that $\cos\theta = \frac{1}{3}$ and find x in terms of a. [5]

21. [9231/s20/33/q1]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O on a smooth horizontal plane. The particle P moves in horizontal circles about O. The tension in the string is $4mg$.

Find, in terms of a and g, the time that P takes to make one complete revolution. [2]

22. [9231/w20/32/q4]

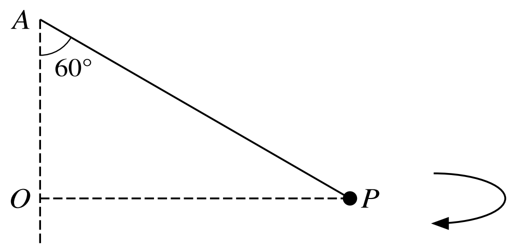
A particle P of mass m is moving in a horizontal circle with angular speed ω on the smooth inner surface of a hemispherical shell of radius r. The angle between the vertical and the normal reaction of the surface on P is θ.

(a) Show that $\cos\theta = \dfrac{g}{\omega^2 r}$. [3]

The plane of the circular motion is at a height x above the lowest point of the shell. When the angular speed is doubled, the plane of the motion is at a height $4x$ above the lowest point of the shell.

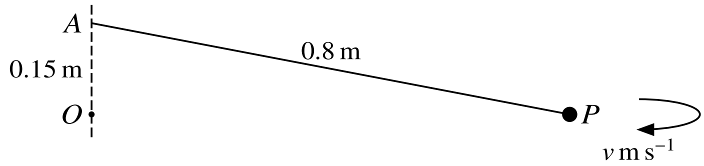
(b) Find x in terms of r. [4]

23. [9709/m19/52/q4]



A particle P of mass 0.3 kg is attached to a fixed point A by a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The particle P moves in a horizontal circle which has centre O. It is given that AO is vertical and that angle OAP is 60° (see diagram). Calculate the speed of P. [6]

24. [9709/s19/51/q1]



A particle P of mass 0.3 kg is attached to a fixed point A by a light inextensible string of length 0.8 m. The fixed point O is 0.15 m vertically below A. The particle P moves with constant speed $v\,\mathrm{m\,s^{-1}}$ in a horizontal circle with centre O (see diagram).

(i) Show that the tension in the string is 16 N. [2]

(ii) Find the value of v. [3]

25. [9709/s19/52/q3]

A particle P of mass 0.4 kg is attached to a fixed point A by a light inextensible string of length 0.5 m. The point A is 0.3 m above a smooth horizontal surface. The particle P moves in a horizontal circle on the surface with constant angular speed 5 rad s^{-1}.

(i) Calculate the tension in the string. [3]

(ii) Find the magnitude of the force exerted by the surface on P. [2]

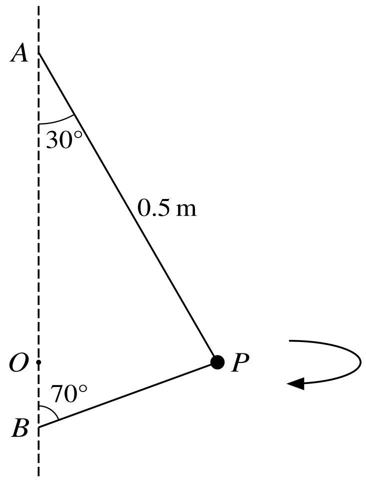
26. [9709/w19/51/q6]

A and B are two fixed points on a vertical axis with A 0.6 m above B. A particle P of mass 0.3 kg is attached to A by a light inextensible string of length 0.5 m. The particle P is attached to B by a light elastic string with modulus of elasticity 46 N. The particle P moves with constant angular speed $8\,\text{rad}\,\text{s}^{-1}$ in a horizontal circle with centre at the mid-point of AB.

(i) Find the speed of P. [2]

(ii) Calculate the tension in the string BP and hence find the natural length of this string. [7]

27. [9709/w19/52/q5]

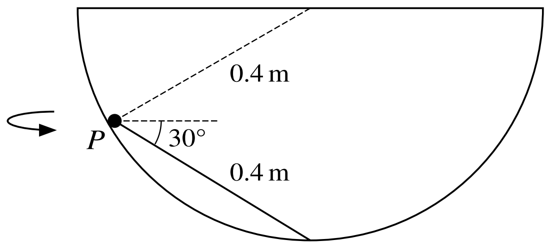


A and B are two fixed points on a vertical axis with A above B. A particle P of mass 0.4 kg is attached to A by a light inextensible string of length 0.5 m. The particle P is attached to B by another light inextensible string. P moves with constant speed in a horizontal circle with centre O between A and B. Angle $BAP = 30°$ and angle $ABP = 70°$ (see diagram).

(i) Given that the tensions in the two strings are equal, find the speed of P. [5]

(ii) Given instead that the angular speed of P is $12\,\text{rad}\,\text{s}^{-1}$, find the tensions in the strings. [5]

28. [9709/m18/52/q5]

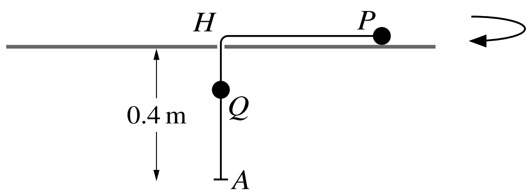


One end of a light inextensible string of length 0.4 m is attached to the lowest point of a hemisphere of radius 0.4 m fixed with its axis vertical. A particle P of mass 0.3 kg is attached to the other end of the string. The string is straight and makes an angle of 30° with the horizontal. P moves on the smooth inner surface of the hemisphere in a horizontal circle (see diagram).

(i) Calculate the smallest possible angular speed of P. [4]

(ii) Given that the greatest possible tension in the string is 5 N, calculate the greatest possible speed of P. [4]

29. [9709/s18/51/q6]

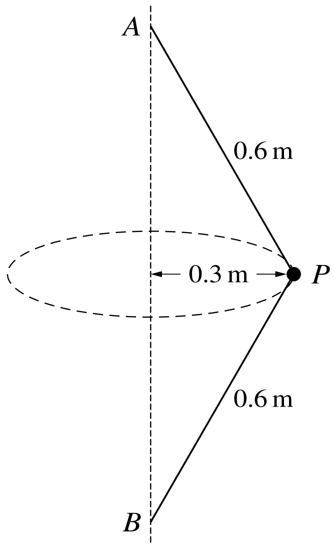


A particle P of mass 0.2 kg is attached to one end of a light inextensible string of length 0.6 m. The other end of the string is attached to a particle Q of mass 0.3 kg. The string passes through a small hole H in a smooth horizontal surface. A light elastic string of natural length 0.3 m and modulus of elasticity 15 N joins Q to a fixed point A which is 0.4 m vertically below H. The particle P moves on the surface in a horizontal circle with centre H (see diagram).

(i) Calculate the greatest possible speed of P for which the elastic string is not extended. [4]

(ii) Find the distance HP given that the angular speed of P is 8 rad s^{-1}. [5]

30. [9709/s18/52/q6]



A particle P of mass 0.2 kg is attached to one end of a light inextensible string of length 0.6 m. The other end of the string is attached to a fixed point A. The particle P is also attached to one end of a second light inextensible string of length 0.6 m, the other end of which is attached to a fixed point B vertically below A. The particle moves in a horizontal circle of radius 0.3 m, which has its centre at the mid-point of AB, with both strings straight (see diagram).

(i) Calculate the least possible angular speed of P. [4]

The string AP will break if its tension exceeds 8 N. The string BP will break if its tension exceeds 5 N.

(ii) Find the greatest possible speed of P. [5]

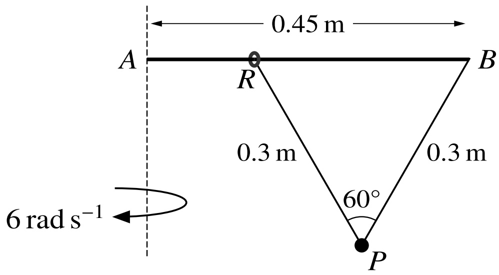
31. [9709/w18/51/q5]

A particle P of mass 0.1 kg is attached to one end of a light inextensible string of length 0.5 m. The other end of the string is attached to a fixed point A. The particle P moves in a circle which has its centre O on a smooth horizontal surface 0.3 m below A. The tension in the string has magnitude T N and the magnitude of the force exerted on P by the surface is R N.

(i) Given that the speed of P is $1.5\ \mathrm{m\,s^{-1}}$, calculate T and R. [4]

(ii) Given instead that $T = R$, calculate the angular speed of P. [4]

32. [9709/w18/52/q7]

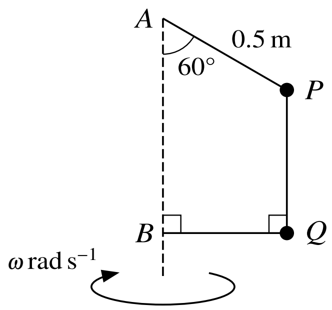


A rough horizontal rod AB of length 0.45 m rotates with constant angular velocity 6 rad s^{-1} about a vertical axis through A. A small ring R of mass 0.2 kg can slide on the rod. A particle P of mass 0.1 kg is attached to the mid-point of a light inextensible string of length 0.6 m. One end of the string is attached to R and the other end of the string is attached to B, with angle $RPB = 60°$ (see diagram). R and P move in horizontal circles as the system rotates. R is in limiting equilibrium.

(i) Show that the tension in the portion PR of the string is 1.66 N, correct to 3 significant figures. [5]

(ii) Find the coefficient of friction between the ring and the rod. [5]

33. [9709/m17/52/q5]



Two particles P and Q have masses 0.4 kg and m kg respectively. P is attached to a fixed point A by a light inextensible string of length 0.5 m which is inclined at an angle of 60° to the vertical. P and Q are joined to each other by a light inextensible vertical string. Q is attached to a fixed point B, which is vertically below A, by a light inextensible string. The string BQ is taut and horizontal. The particles rotate in horizontal circles about an axis through A and B with constant angular speed ω rad s^{-1} (see diagram). The tension in the string joining P and Q is 1.5 N.

(i) Find the tension in the string AP and the value of ω. [4]

(ii) Find m and the tension in the string BQ. [3]

34. [9709/s17/51/q6]

A particle P of mass 0.15 kg is attached to one end of a light elastic string of natural length 0.4 m and modulus of elasticity 12 N. The other end of the string is attached to a fixed point A. The particle P moves in a horizontal circle which has its centre vertically below A, with the string inclined at $\theta°$ to the vertical and $AP = 0.5$ m.

(i) Find the angular speed of P and the value of θ. [5]

(ii) Calculate the difference between the elastic potential energy stored in the string and the kinetic energy of P. [4]

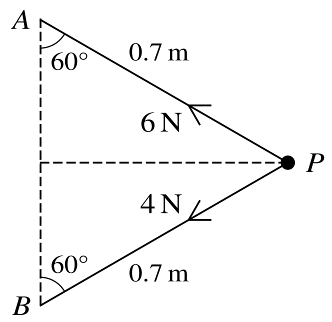
35. [9709/s17/52/q1]

A particle P of mass 0.2 kg moves with speed $4\,\mathrm{m\,s^{-1}}$ and angular speed $5\,\mathrm{rad\,s^{-1}}$ in a horizontal circle on a smooth surface. P is attached to one end of a light elastic string of natural length 0.6 m. The other end of the string is attached to the point on the surface which is the centre of the circular motion of P.

(i) Find the radius of this circle. [1]

(ii) Find the modulus of elasticity of the string. [4]

36. [9709/s17/52/q2]

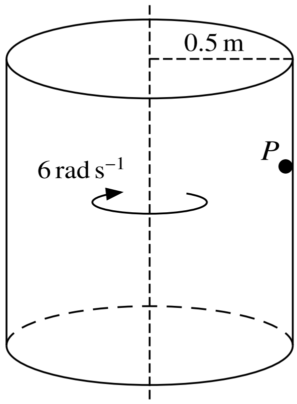


The ends of two light inextensible strings of length 0.7 m are attached to a particle P. The other ends of the strings are attached to two fixed points A and B which lie in the same vertical line with A above B. The particle P moves in a horizontal circle which has its centre at the mid-point of AB. Both strings are inclined at 60° to the vertical. The tension in the string attached to A is 6 N and the tension in the string attached to B is 4 N (see diagram).

(i) Find the mass of P. [2]

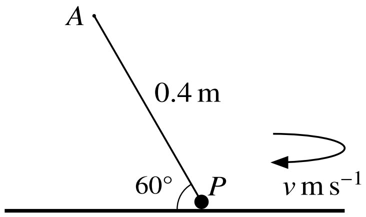
(ii) Calculate the speed of P. [3]

37. [9709/w17/51/q1]



A hollow cylinder with a rough inner surface has radius 0.5 m. A particle P of mass 0.4 kg is in contact with the inner surface of the cylinder. The particle and cylinder rotate together with angular speed 6 rad s^{-1} about the vertical axis of the cylinder, so that the particle moves in a horizontal circle (see diagram). Given that P is about to slip downwards, find the coefficient of friction between P and the surface of the cylinder. [4]

38. [9709/w17/51/q3]



One end of a light inextensible string of length 0.4 m is attached to a fixed point A which is above a smooth horizontal surface. A particle P of mass 0.6 kg is attached to the other end of the string. P moves in a circle on the surface with constant speed $v\,\text{m s}^{-1}$, with the string taut and making an angle of 60° with the horizontal (see diagram).

(i) Given that $v = 0.5$, calculate the magnitude of the force that the surface exerts on P. [4]

(ii) Find the greatest possible value of v for which P remains in contact with the surface. [3]

39. [9709/w17/52/q6]

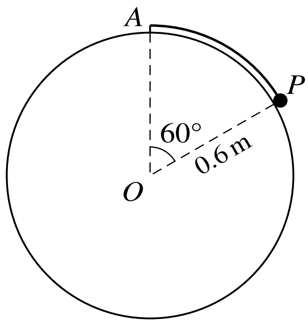
One end of a light elastic string of natural length 0.4 m and modulus of elasticity 8 N is attached to a fixed point O on a smooth horizontal plane. The other end of the string is attached to a particle P of mass 0.2 kg which moves on the plane in a circular path with centre O. The speed of P is $v\,\mathrm{m\,s^{-1}}$ and the extension of the string is x m.

(i) Given that $v = 2.5$, find x. [4]

It is given instead that the kinetic energy of P is twice the elastic potential energy stored in the string.

(ii) Form two simultaneous equations and hence find x and v. [5]

40. [9709/m16/52/q7]



One end of a light inextensible string is attached to the highest point A of a solid fixed sphere with centre O and radius 0.6 m. The other end of the string is attached to a particle P of mass 0.2 kg which rests in contact with the smooth surface of the sphere. The angle $AOP = 60°$ (see diagram). The sphere exerts a contact force of magnitude R N on P and the tension in the string is T N.

(i) By resolving vertically, show that $R + (\sqrt{3})T = 4$. [2]

P is now set in motion, and moves with angular speed ω rad s^{-1} in a horizontal circle on the surface of the sphere.

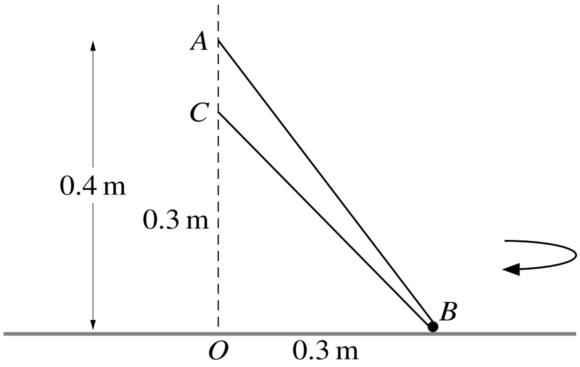
(ii) Find an equation involving R, T and ω. [2]

(iii) Hence

(a) calculate R when $\omega = 2$, [2]

(b) find the greatest possible value of T and the corresponding speed of P. [4]

41. [9709/s16/51/q6]



A light inextensible string passes through a small smooth bead B of mass 0.4 kg. One end of the string is attached to a fixed point A 0.4 m above a fixed point O on a smooth horizontal surface. The other end of the string is attached to a fixed point C which is vertically below A and 0.3 m above the surface. The bead moves with constant speed on the surface in a circle with centre O and radius 0.3 m (see diagram).

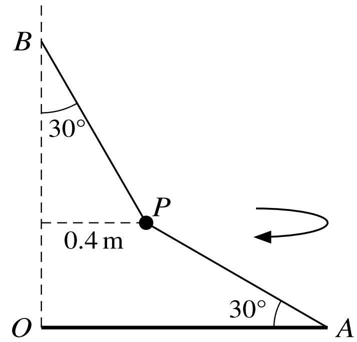
(i) Given that the tension in the string is 2 N, calculate

(a) the angular speed of the bead, [3]

(b) the magnitude of the contact force exerted on the bead by the surface. [2]

(ii) Given instead that the bead is about to lose contact with the surface, calculate the speed of the bead. [4]

42. [9709/s16/52/q6]



OA is a rod which rotates in a horizontal circle about a vertical axis through O. A particle P of mass 0.2 kg is attached to the mid-point of a light inextensible string. One end of the string is attached to the rod at A and the other end of the string is attached to a point B on the axis. It is given that $OA = OB$, angle OAP = angle $OBP = 30°$, and P is 0.4 m from the axis. The rod and the particle rotate together about the axis with P in the plane OAB (see diagram).

(i) Calculate the tensions in the two parts of the string when the speed of P is $1.2\,\mathrm{m\,s^{-1}}$. [6]

The angular speed of the rod is increased to $5\,\mathrm{rad\,s^{-1}}$, and it is given that the system now rotates with angle OAP = angle $OBP = 60°$.

(ii) Show that the tension in the part AP of the string is zero. [6]

43. [9709/w16/51/q1]

A particle P of mass 0.3 kg moves in a circle with centre O on a smooth horizontal surface. P is attached to O by a light elastic string of modulus of elasticity 12 N and natural length l m. The speed of P is $4\,\mathrm{m\,s^{-1}}$, and the radius of the circle in which it moves is $2l$ m. Calculate l. [4]

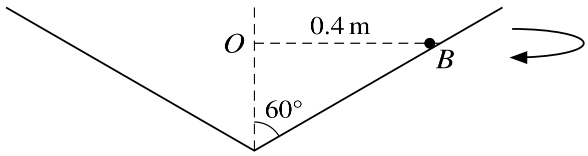
44. [9709/w16/51/q5]

A small ball B of mass 0.4 kg moves in a horizontal circle with centre O and radius 0.6 m on a smooth horizontal surface. One end of a light inextensible string is attached to B; the other end of the string is attached to a fixed point 0.45 m vertically above O.

(i) Given that the tension in the string is 5 N, calculate the speed of B. [3]

(ii) Find the greatest possible tension in the string for the motion, and the corresponding angular speed of B. [4]

45. [9709/w16/52/q7]



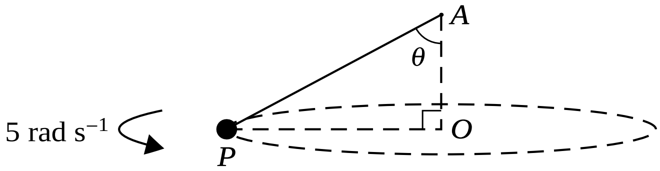
A small ball B of mass 0.5 kg moves in a horizontal circle with centre O and radius 0.4 m on the smooth inner surface of a hollow cone fixed with its vertex down. The axis of the cone is vertical and the semi-vertical angle is 60° (see diagram).

(i) Show that the magnitude of the force exerted by the cone on B is 5.77 N, correct to 3 significant figures, and calculate the angular speed of B. [4]

One end of a light elastic string of natural length 0.45 m and modulus of elasticity 36 N is attached to B. The other end of the string is attached to the point on the axis 0.3 m above O. The ball B again moves on the surface of the cone in the same horizontal circle as before.

(ii) Calculate the speed of B. [6]

46. [9709/s15/51/q3]



One end of a light inextensible string is attached to a fixed point A and the other end of the string is attached to a particle P. The particle P moves with constant angular speed $5\,\text{rad}\,\text{s}^{-1}$ in a horizontal circle which has its centre O vertically below A. The string makes an angle θ with the vertical (see diagram). The tension in the string is three times the weight of P.

(i) Show that the length of the string is $1.2\,\text{m}$. [3]

(ii) Find the speed of P. [4]

47. [9709/s15/52/q1]

A particle P of mass 0.6 kg is on the rough surface of a horizontal disc with centre O. The distance OP is 0.4 m. The disc and P rotate with angular speed $3\,\text{rad}\,\text{s}^{-1}$ about a vertical axis which passes through O. Find the magnitude of the frictional force which the disc exerts on the particle, and state the direction of this force. [3]

48. [9709/s15/52/q4]

One end of a light inextensible string of length 0.5 m is attached to a fixed point A. The other end of the string is attached to a particle P of weight 6 N. Another light inextensible string of length 0.5 m connects P to a fixed point B which is 0.8 m vertically below A. The particle P moves with constant speed in a horizontal circle with centre at the mid-point of AB. Both strings are taut.

(i) Calculate the speed of P when the tension in the string BP is 2 N. [5]

(ii) Show that the angular speed of P must exceed $5\,\text{rad}\,\text{s}^{-1}$. [3]

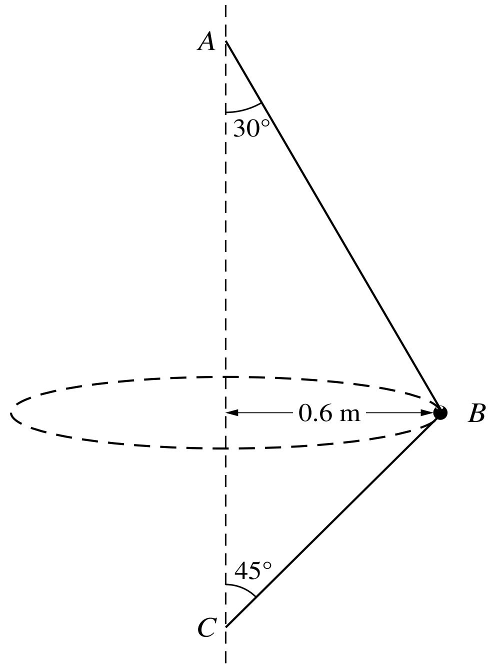
49. [9709/s15/53/q7]

A particle P of mass 0.7 kg is attached to one end of a light inextensible string of length 0.5 m. The other end of the string is attached to a fixed point A which is h m above a smooth horizontal surface. P moves in contact with the surface with uniform circular motion about the point on the surface which is vertically below A.

(i) Given that $h = 0.14$, find an inequality for the angular speed of P. [4]

(ii) Given instead that the magnitude of the force exerted by the surface on P is 1.4 N and that the speed of P is 2.5 m s^{-1}, calculate the tension in the string and the value of h. [7]

50. [9709/w15/51/q4]



One end of a light inextensible string is attached to a fixed point A. The string passes through a smooth bead B of mass 0.3 kg and the other end of the string is attached to a fixed point C vertically below A. The bead B moves with constant speed in a horizontal circle of radius 0.6 m which has its centre between A and C. The string makes an angle of 30° with the vertical at A and an angle of 45° with the vertical at C (see diagram).

(i) Calculate the speed of B. [5]

The lower end of the string is detached from C, and B is now attached to this end of the string. The other end of the string remains attached to A. The bead is set in motion so that it moves with angular speed 3 rad s^{-1} in a horizontal circle which has its centre vertically below A.

(ii) Calculate the tension in the string. [3]

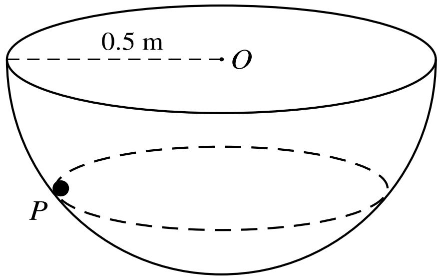
51. [9709/w15/53/q2]

One end of a light inextensible string of length 0.5 m is attached to a fixed point A. A particle P of mass 0.2 kg is attached to the other end of the string. P moves with constant speed in a horizontal circle with centre O which is 0.4 m vertically below A.

(i) Show that the tension in the string is 2.5 N. [2]

(ii) Find the speed of P. [3]

52. [9709/w15/53/q4]



A particle P of mass 0.4 kg moves with constant speed in a horizontal circle on the smooth inner surface of a fixed hollow hemisphere with centre O and radius 0.5 m (see diagram).

(i) Given that the speed of the particle is $4\,\text{m}\,\text{s}^{-1}$ and its angular speed is $10\,\text{rad}\,\text{s}^{-1}$, calculate the angle between OP and the vertical. [2]

(ii) Given instead that the magnitude of the force exerted on P by the hemisphere is 6 N, calculate

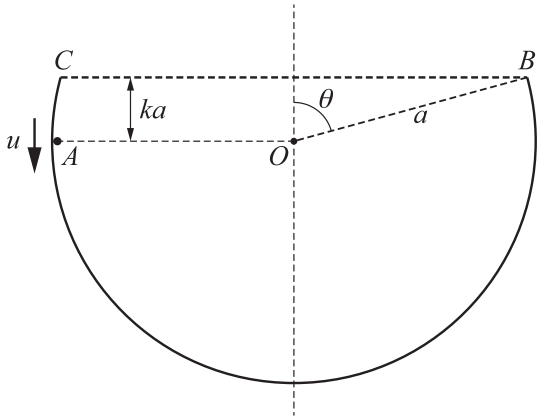
(a) the angle between OP and the vertical, [2]

(b) the angular speed of P. [3]

Chapter 4

Circular motion (vertical)

1. [9231/s25/31/q7]



A fixed hollow sphere has radius a and centre O. The points A, B and C lie on the inner surface of the sphere with OA and BC horizontal. A portion of the sphere has been removed by a horizontal cut through points B and C at a vertical distance ka above the centre of the sphere, where k is a positive constant and $k < 1$. The points O, A, B and C all lie in the same vertical plane. OB makes an angle θ with the upward vertical through O (see diagram).

A particle P of mass m is free to move on the smooth inner surface of the sphere. The particle P is projected vertically downwards from A with speed u and begins to move in a vertical circle.

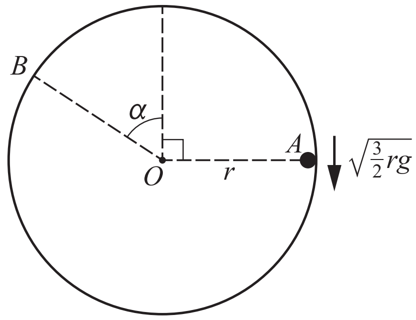
(a) In the case where $u = \sqrt{\frac{6}{5}ga}$, the reaction on P at B is half the reaction on P at A.

Find the value of k. [5]

(b) Find an expression for u, in terms of a and g, in the case that the particle just reaches B. [1]

(c) Find an expression for u, in terms of a and g, in the case that the particle passes through B and in its subsequent motion reaches C. [4]

2. [9231/s25/33/q5]



A hollow cylinder of radius r is fixed with its axis horizontal. Points A, B and O are in the same vertical plane perpendicular to the axis of the cylinder, with A and B on the smooth inner surface and O on the axis. OA and OB make angles 90° and α respectively with the upward vertical through O, with A and B on opposite sides of the vertical. A particle of mass m is projected vertically downwards from point A with speed $\sqrt{\frac{3}{2}rg}$ and moves in a vertical circle inside the cylinder (see diagram). The particle loses contact with the cylinder at point B.

(a) Find the value of α. [4]

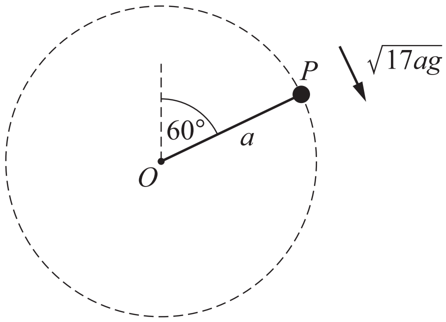
(b) In the subsequent motion find, in terms of r, the greatest height above O reached by the particle. [4]

3. [9231/s25/34/q7]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in complete vertical circles with centre O, with the string taut. When the string makes an angle θ with the downward vertical through O the speed of P is $\sqrt{4ag}$. The ratio of the greatest and least tensions in the string during the motion is 11 : 1.

Find the value of $\cos\theta$. [8]

4. [9231/w25/31/q6]



A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. Initially P is held with the string taut and making an angle of 60° with the upward vertical through O. The particle P is projected perpendicular to the string in a downwards direction with speed $\sqrt{17ag}$. It then starts to move along a circular path in a vertical plane with centre O (see diagram). At the lowest point of its path, vertically below O, the particle P collides with a stationary particle Q.

(a) Find, in terms of a and g, an expression for the speed of P immediately before the collision with Q. [2]

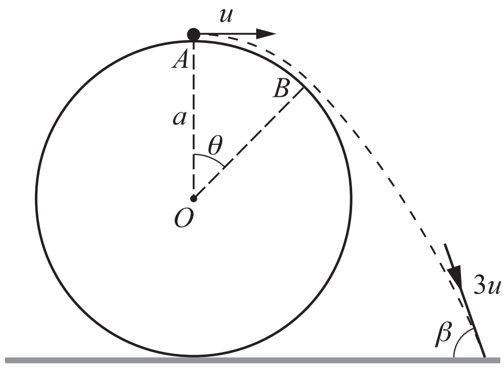
As a result of the collision, P rebounds and moves back along a circular path with centre O. The string becomes slack when P reaches the point on the circle vertically above O.

(b) Find, in terms of a and g, an expression for the speed of P immediately after the collision with Q. [3]

The mass of particle Q is km and the collision between P and Q is perfectly elastic.

(c) Find the value of k. [3]

5. [9231/w25/32/q6]

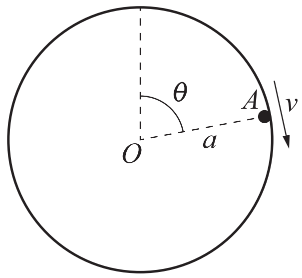


A fixed smooth sphere with radius a and centre O rests on horizontal ground. A particle is projected horizontally from the highest point, A, of the sphere with speed u. The particle begins to move in a vertical circle along the surface of the sphere. The particle loses contact with the sphere at the point B, where the angle AOB is θ.

After leaving the surface of the sphere, the particle moves freely under gravity before striking the horizontal ground with speed $3u$ at an angle β to the horizontal (see diagram).

Find the value of β. [8]

6. [9231/w25/34/q4]



A fixed smooth spherical shell has centre O and radius a. A particle of mass m moves in complete vertical circles on the smooth inner surface of the shell, where the plane of the circular motion is vertical and passes through O. The particle has speed v when it is at point A, where OA makes an angle θ with the upward vertical through O, and $\cos\theta = \frac{1}{18}$ (see diagram).

(a) Show that $v \geqslant \frac{1}{3}\sqrt{26ag}$. [5]

It is given that $v = \frac{1}{3}\sqrt{26ag}$.

(b) Find, in terms of m and g, an expression for the greatest possible value of the normal reaction between the shell and the particle. [3]

7. [9231/s24/31/q7]

A smooth sphere with centre O and of radius a is fixed to a horizontal plane. A particle P of mass m is projected horizontally from the highest point of the sphere with speed u, so that it begins to move along the surface of the sphere. The particle P loses contact with the sphere at the point Q on the sphere, where OQ makes an angle θ with the upward vertical through O.

(a) Show that $\cos\theta = \dfrac{u^2 + 2ag}{3ag}$. [4]

It is given that $\cos\theta = \frac{5}{6}$.

(b) Find, in terms of a and g, an expression for the vertical component of the velocity of P just before it hits the horizontal plane to which the sphere is fixed. [3]

(c) Find an expression for the time taken by P to fall from Q to the plane. Give your answer in the form $k\sqrt{\frac{a}{g}}$, stating the value of k correct to 3 significant figures. [2]

8. [9231/s24/33/q3]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When the particle is hanging vertically below O, it is projected horizontally with speed u so that it begins to move along a circular path. When P is at the lowest point of its motion, the tension in the string is T. When OP makes an angle θ with the upward vertical, the tension in the string is S.

(a) Show that $S = T - 3mg(1 + \cos\theta)$. [5]

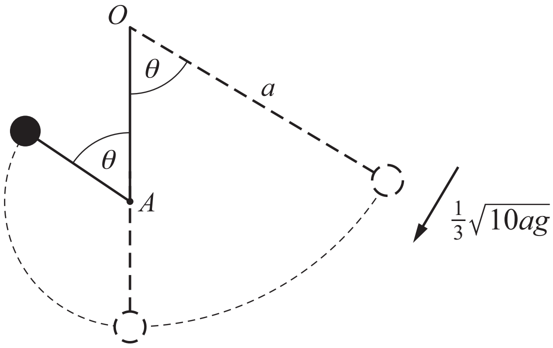
(b) Given that $u = \sqrt{4ag}$, find the value of $\cos\theta$ when the string goes slack. [2]

9. [9231/w24/31/q2]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A with the string taut. It is given that OA makes an angle θ with the downward vertical through O, where $\tan\theta = \frac{3}{4}$. The particle P is projected perpendicular to OA in an upwards direction with speed $\sqrt{5ag}$, and it starts to move along a circular path in a vertical plane. When P is at the point B, where angle AOB is a right angle, the tension in the string is T.

Find T in terms of m and g. [5]

10. [9231/w24/32/q6]



A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held with the string taut and the string makes an angle θ with the downward vertical through O. The particle P is projected at right angles to the string with speed $\frac{1}{3}\sqrt{10ag}$ and begins to move downwards along a circular path. When the string is vertical, it strikes a small smooth peg at the point A which is vertically below O. The circular path and the point A are in the same vertical plane. After the string strikes the peg, the particle P begins to move in a vertical circle with centre A. When the string makes an angle θ with the upward vertical through A the string becomes slack (see diagram). The distance of A below O is $\frac{5}{9}a$.

(a) Find the value of $\cos\theta$. [6]

(b) Find the ratio of the tensions in the string immediately before and immediately after it strikes the peg. [4]

11. [9231/s23/31/q3]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle θ with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed u. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where angle AOB is 90° and the speed of P is $\sqrt{\frac{4}{5}ag}$.

(a) Find the value of $\sin\theta$. [2]

(b) Find, in terms of m and g, the tension in the string when P is at A. [5]

12. [9231/s23/33/q1]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle α with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed $\sqrt{3ag}$. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where OB makes an angle θ with the upward vertical.

Given that $\cos\alpha = \frac{4}{5}$, find the value of $\cos\theta$. [4]

13. [9231/w23/31/q6]

A particle P of mass m is attached to one end of a light inextensible rod of length $3a$. An identical particle Q is attached to the other end of the rod. The rod is smoothly pivoted at a point O on the rod, where $OQ = x$. The system, of rod and particles, rotates about O in a vertical plane.

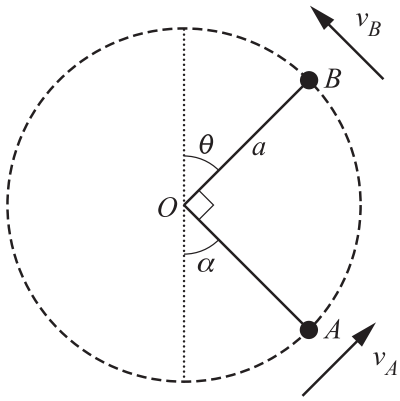
At an instant when the rod is vertical, with P above Q, the particle P is moving horizontally with speed u. When the rod has turned through an angle of 60° from the vertical, the speed of P is $2\sqrt{ag}$, and the tensions in the two parts of the rod, OP and OQ, have equal magnitudes.

(a) Show that the speed of Q when the rod has turned through an angle of 60° from the vertical is $\dfrac{2x}{3a-x}\sqrt{ag}$. [2]

(b) Find x in terms of a. [5]

(c) Find u in terms of a and g. [4]

14. [9231/w23/32/q5]



A bead of mass m moves on a smooth circular wire, with centre O and radius a, in a vertical plane. The bead has speed v_A when it is at the point A where OA makes an angle α with the downward vertical through O, and $\cos\alpha = \frac{3}{5}$. Subsequently the bead has speed v_B at the point B, where OB makes an angle θ with the upward vertical through O. Angle AOB is a right angle (see diagram). The reaction of the wire on the bead at B is in the direction OB and has magnitude equal to $\frac{1}{6}$ of the magnitude of the reaction when the bead is at A.

(a) Find, in terms of m and g, the magnitude of the reaction at B. [6]

(b) Given that $v_A = \sqrt{kag}$, find the value of k. [2]

15. [9231/s22/31/q2]

One end of a light inextensible string of length a is attached to a fixed point O. A particle of mass m is attached to the other end of the string. The particle is held at the point A with the string taut. The angle between OA and the downward vertical is equal to α, where $\cos\alpha = \frac{4}{5}$. The particle is projected from A, perpendicular to the string in an upwards direction, with a speed $\sqrt{3ga}$. It then moves along a circular path in a vertical plane. The string first goes slack when it makes an angle θ with the upward vertical through O.

Find the value of $\cos\theta$. [5]

16. [9231/s22/33/q4]

One end of a light inextensible string of length a is attached to a fixed point O. A particle of mass m is attached to the other end of the string and is held with the string taut at the point A. At A the string makes an angle θ with the upward vertical through O. The particle is projected perpendicular to the string in a downward direction from A with a speed u. It moves along a circular path in the vertical plane.

When the string makes an angle α with the downward vertical through O, the speed of the particle is $2u$ and the magnitude of the tension in the string is 10 times its magnitude at A.

It is given that $u = \sqrt{\frac{2}{3}ga}$.

(a) Find, in terms of m and g, the magnitude of the tension in the string at A. [6]

(b) Find the value of $\cos\alpha$. [2]

17. [9231/w22/31/q5]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The string is held taut with OP horizontal. The particle P is projected vertically downwards with speed $\sqrt{\frac{1}{3}ag}$ and starts to move in a vertical circle. P passes through the lowest point of the circle and reaches the point Q where OQ makes an angle θ with the downward vertical. At Q the speed of P is $\sqrt{kag}$ and the tension in the string is $\frac{11}{6}mg$.

(a) Find the value of k and the value of $\cos\theta$. [4]

At Q the particle P becomes detached from the string.

(b) In the subsequent motion, find the greatest height reached by P above the level of the lowest point of the circle. [4]

18. [9231/w22/32/q1]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The string is held taut with OP making an angle α with the downward vertical, where $\cos\alpha = \frac{2}{3}$. The particle P is projected perpendicular to OP in an upwards direction with speed $\sqrt{3ag}$. It then starts to move along a circular path in a vertical plane.

Find the cosine of the angle between the string and the upward vertical when the string first becomes slack. [4]

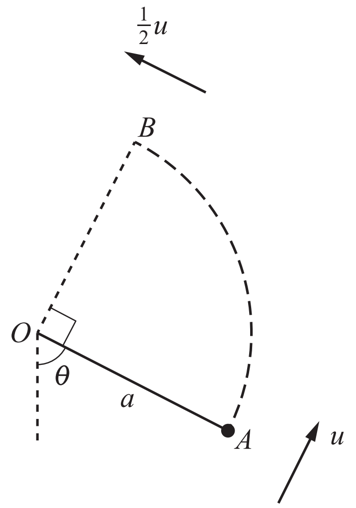
19. [9231/s21/31/q5]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle completes vertical circles with centre O. The points A and B are on the path of P, both on the same side of the vertical through O. OA makes an angle θ with the downward vertical through O and OB makes an angle θ with the upward vertical through O.

The speed of P when it is at A is u and the speed of P when it is at B is $\sqrt{ag}$. The tensions in the string at A and B are T_A and T_B respectively. It is given that $T_A = 7T_B$.

Find the value of θ and find an expression for u in terms of a and g. [8]

20. [9231/s21/33/q4]



A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is initially held with the string taut at the point A, where OA makes an angle θ with the downward vertical through O. The particle is then projected with speed u perpendicular to OA and begins to move upwards in part of a vertical circle. The string goes slack when the particle is at the point B where angle AOB is a right angle. The speed of the particle when it is at B is $\frac{1}{2}u$ (see diagram).

Find the tension in the string at A, giving your answer in terms of m and g. [8]

21. [9231/w21/31/q6]

A particle P, of mass m, is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P moves in complete vertical circles about O with the string taut. The points A and B are on the path of P with AB a diameter of the circle. OA makes an angle θ with the downward vertical through O and OB makes an angle θ with the upward vertical through O. The speed of P when it is at A is $\sqrt{5ag}$.

The ratio of the tension in the string when P is at A to the tension in the string when P is at B is 9 : 5.

(a) Find the value of $\cos\theta$. [6]

(b) Find, in terms of a and g, the greatest speed of P during its motion. [2]

22. [9231/w21/32/q7]

One end of a light inextensible string of length a is attached to a fixed point O. The other end of the string is attached to a particle P of mass m. The particle P is held vertically below O with the string taut and then projected horizontally. When the string makes an angle of 60° with the upward vertical, P becomes detached from the string. In its subsequent motion, P passes through the point A which is a distance a vertically above O.

(a) The speed of P when it becomes detached from the string is V. Use the equation of the trajectory of a projectile to find V in terms of a and g. [4]

(b) Find, in terms of m and g, the tension in the string immediately after P is initially projected horizontally. [4]

23. [9231/s20/31/q7]

A hollow cylinder of radius a is fixed with its axis horizontal. A particle P, of mass m, moves in part of a vertical circle of radius a and centre O on the smooth inner surface of the cylinder. The speed of P when it is at the lowest point A of its motion is $\sqrt{\frac{7}{2}ga}$.

The particle P loses contact with the surface of the cylinder when OP makes an angle θ with the upward vertical through O.

(a) Show that $\theta = 60°$. [5]

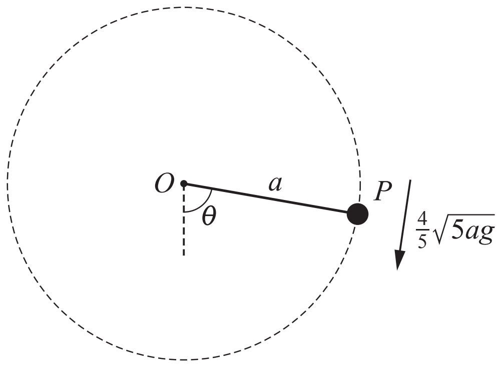
(b) Show that in its subsequent motion P strikes the cylinder at the point A. [5]

24. [9231/s20/33/q3]

A particle Q of mass m is attached to a fixed point O by a light inextensible string of length a. The particle moves in complete vertical circles about O. The points A and B are on the path of Q with AB a diameter of the circle. OA makes an angle of 60° with the downward vertical through O and OB makes an angle of 60° with the upward vertical through O. The speed of Q when it is at A is $2\sqrt{ag}$.

Given that T_A and T_B are the tensions in the string at A and B respectively, find the ratio $T_A : T_B$. [6]

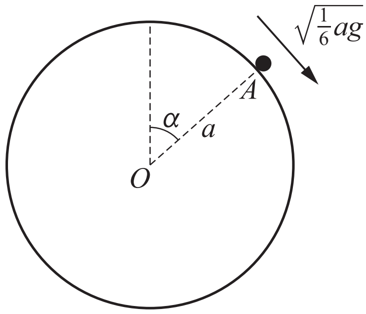
25. [9231/w20/31/q2]



A particle P is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held with the string taut and making an angle θ with the downward vertical. The particle P is then projected with speed $\frac{4}{5}\sqrt{5ag}$ perpendicular to the string and just completes a vertical circle (see diagram).

Find the value of $\cos\theta$. [5]

26. [9231/w20/32/q1]



A fixed smooth solid sphere has centre O and radius a. A particle of mass m is projected downwards with speed $\sqrt{\frac{1}{6}ag}$ from the point A on the surface of the sphere, where OA makes an angle α with the upward vertical through O (see diagram). The particle moves in part of a vertical circle on the surface of the sphere. It loses contact with the sphere at the point B, where OB makes an angle β with the upward vertical through O.

Given that $\cos\alpha = \frac{2}{3}$, find the value of $\cos\beta$. [5]

27. [9231/s19/21/q11e]

A particle P, of mass m, is able to move in a vertical circle on the smooth inner surface of a sphere with centre O and radius a. Points A and B are on the inner surface of the sphere and AOB is a horizontal diameter. Initially, P is projected vertically downwards with speed $\sqrt{\left(\frac{21}{2}ag\right)}$ from A and begins to move in a vertical circle. At the lowest point of its path, vertically below O, the particle P collides with a stationary particle Q, of mass $4m$, and rebounds. The speed acquired by Q, as a result of the collision, is just sufficient for it to reach the point B.

(i) Find the speed of P and the speed of Q immediately after their collision. [7]

In its subsequent motion, P loses contact with the inner surface of the sphere at the point D, where the angle between OD and the upward vertical through O is θ.

(ii) Find $\cos\theta$. [5]

28. [9231/s19/23/q2]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is moving in a complete vertical circle about O. The points A and B are on the circle, at opposite ends of a diameter, and such that OA makes an acute angle α with the upward vertical through O. The speed of P as it passes through A is $\frac{3}{2}\sqrt{(ag)}$. The tension in the string when P is at B is four times the tension in the string when P is at A.

(i) Show that $\cos\alpha = \frac{3}{4}$. [6]

(ii) Find the tension in the string when P is at B. [2]

29. [9231/w19/21/q4]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O and P is held with the string taut and horizontal. The particle P is projected vertically downwards with speed $\sqrt{(2ag)}$ so that it begins to move along a circular path. The string becomes slack when OP makes an angle θ with the upward vertical through O.

(i) Show that $\cos\theta = \frac{2}{3}$. [5]

(ii) Find the greatest height, above the horizontal through O, reached by P in its subsequent motion. [4]

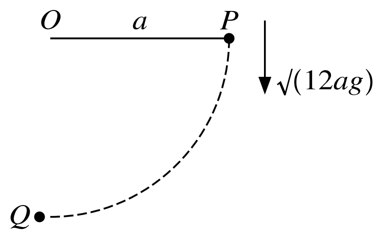
30. [9231/s18/21/q11e]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held so that the string is taut, with OP horizontal. The particle is projected downwards with speed $\sqrt{\left(\frac{2}{5}ag\right)}$ and begins to move in a vertical circle. The string breaks when its tension is equal to $\frac{11}{5}mg$.

(i) Show that the string breaks when OP makes an angle θ with the downward vertical through O, where $\cos\theta = \frac{3}{5}$. Find the speed of P at this instant. [6]

(ii) For the subsequent motion after the string breaks, find the distance OP when the particle P is vertically below O. [6]

31. [9231/s18/23/q5]



A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held with the string taut and horizontal. It is projected downwards with speed $\sqrt{(12ag)}$. At the lowest point of its motion, P collides directly with a particle Q of mass km which is at rest (see diagram). In the collision, P and Q coalesce. The tension in the string immediately after the collision is half of its value immediately before the collision. Find the possible values of k. [11]

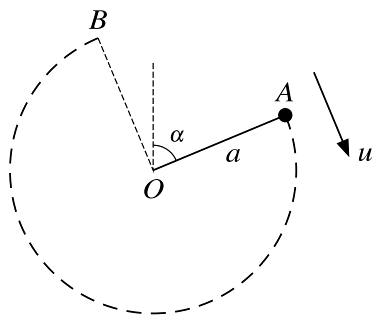
32. [9231/w18/21/q11e]

A particle P of mass m is free to move on the smooth inner surface of a fixed hollow sphere of radius a. The centre of the sphere is O and the point C is on the inner surface of the sphere, vertically below O. The points A and B on the inner surface of the sphere are the ends of a diameter of the sphere. The diameter AOB makes an acute angle α with the vertical, where $\cos\alpha = \frac{4}{5}$, with A below the horizontal level of B. The particle is projected from A with speed u, and moves along the inner surface of the sphere towards C. The normal reaction forces on the particle at A and C are in the ratio 8 : 9.

(i) Show that $u^2 = 4ag$. [6]

(ii) Determine whether P reaches B without losing contact with the inner surface of the sphere. [6]

33. [9231/w18/22/q3]

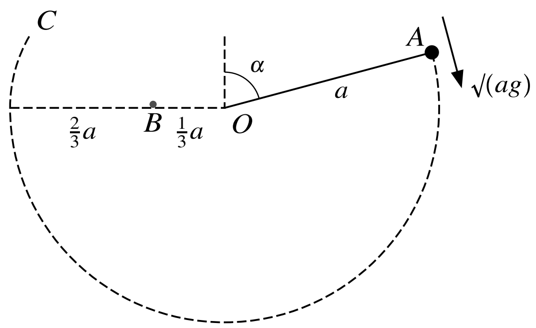


A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is such that $OA = a$ and OA makes an angle α with the upward vertical, where $\tan\alpha = \frac{12}{5}$. The particle is projected downwards from A with speed u perpendicular to the string and moves in a vertical plane (see diagram). The string becomes slack after the string has rotated through 270° from its initial position, with the particle now at the point B.

(i) Show that $u^2 = 2ag$. [5]

(ii) Find the maximum tension in the string as the particle moves from A to B. [4]

34. [9231/s17/21/q5]



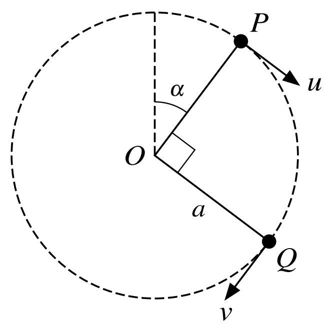
A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is such that $OA = a$ and OA makes an angle α with the upward vertical through O. The particle is held at A and then projected downwards with speed $\sqrt{(ag)}$ so that it begins to move in a vertical circle with centre O. There is a small smooth peg at the point B which is at the same horizontal level as O and at a distance $\frac{1}{3}a$ from O on the opposite side of O to A (see diagram).

(i) Show that, when the string first makes contact with the peg, the speed of the particle is $\sqrt{\left(ag(1 + 2\cos\alpha)\right)}$. [2]

The particle now begins to move in a vertical circle with centre B. When the particle is at the point C where angle $CBO = 150°$, the tension in the string is the same as it was when the particle was at the point A.

(ii) Find the value of $\cos\alpha$. [10]

35. [9231/s17/23/q5]



A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is moving in complete vertical circles with the string taut. When the particle is at the point P, where OP makes an angle α with the upward vertical through O, its speed is u. When the particle is at the point Q, where angle $QOP = 90°$, its speed is v (see diagram). It is given that $\cos\alpha = \frac{4}{5}$.

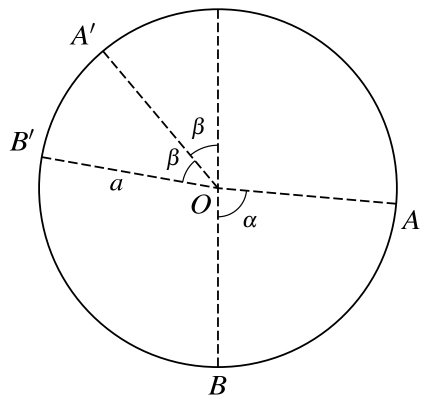
(i) Show that $v^2 = u^2 + \frac{14}{5}ag$. [2]

The tension in the string when the particle is at Q is twice the tension in the string when the particle is at P.

(ii) Obtain another equation relating u^2, v^2, a and g, and hence find u in terms of a and g. [5]

(iii) Find the least tension in the string during the motion. [3]

36. [9231/w17/21/q11e]



A particle P of mass m is free to move on the smooth inner surface of a fixed hollow sphere of radius a. The centre of the sphere is O. The points A and A' are on the inner surface of the sphere, on opposite sides of the vertical through O; the radius OA makes an angle α with the downward vertical and the radius OA' makes an angle β with the upward vertical. The point B is on the inner surface of the sphere, vertically below O. The point B' is on the inner surface of the sphere and such that OB' makes an angle 2β with the upward vertical through O (see diagram). It is given that $\cos\alpha = \frac{1}{16}$.

(i) P is projected from A with speed u along the surface of the sphere downwards towards B. Subsequently it loses contact with the sphere at A'. Show that $u^2 = \frac{1}{8}ag(1 + 24\cos\beta)$. [5]

(ii) P is now projected from B with speed u along the surface of the sphere towards B'. Subsequently it loses contact with the sphere at B'. Find $\cos\beta$. [6]

(iii) In part **(i)**, the reaction of the sphere on P when it is initially projected at A is R. Find R in terms of m and g. [3]

37. [9231/s16/21/q4]

A particle P is at rest at the lowest point on the smooth inner surface of a hollow sphere with centre O and radius a. The particle is projected horizontally with speed u and begins to move in a vertical circle on the inner surface of the sphere. The particle loses contact with the sphere at the point A, where OA makes an angle θ with the upward vertical through O. Given that the speed of P at A is $\sqrt{\left(\frac{3}{5}ag\right)}$, find u in terms of a and g. [5]

Find, in terms of a, the greatest height above the level of O achieved by P in its subsequent motion. (You may assume that P achieves its greatest height before it makes any further contact with the sphere.) [5]

38. [9231/s16/23/q1]

A particle P is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in complete vertical circles with centre O. The tension in the string when P is at its lowest point is twice the tension in the string when P is at its highest point. Find, in terms of a and g, the greatest speed of P during the motion. [6]

39. [9231/w16/21/q4]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held vertically above O with the string taut and then projected horizontally with speed $\sqrt{\left(\frac{13}{3}ag\right)}$. It begins to move in a vertical circle with centre O. When P is at its lowest point, it collides with a stationary particle of mass λm. The two particles coalesce.

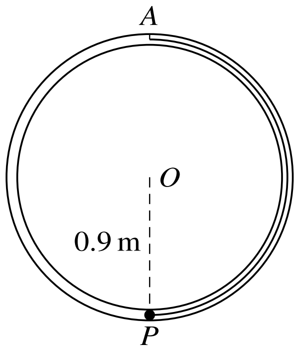
(i) Show that the speed of the combined particle immediately after the impact is $\dfrac{5}{\lambda+1}\sqrt{\left(\frac{1}{3}ag\right)}$. [4]

In the subsequent motion, the string becomes slack when the combined particle is at a height of $\frac{1}{3}a$ above the level of O.

(ii) Find the value of λ. [6]

(iii) Find, in terms of m and g, the instantaneous change in the tension in the string as a result of the collision. [4]

40. [9709/w16/51/q6]



The diagram shows a smooth narrow tube formed into a fixed vertical circle with centre O and radius 0.9 m. A light elastic string with modulus of elasticity 8 N and natural length 1.2 m has one end attached to the highest point A on the inside of the tube. The other end of the string is attached to a particle P of mass 0.2 kg. The particle is released from rest at the lowest point on the inside of the tube. By considering energy, calculate

(i) the speed of P when it is at the same horizontal level as O, [4]

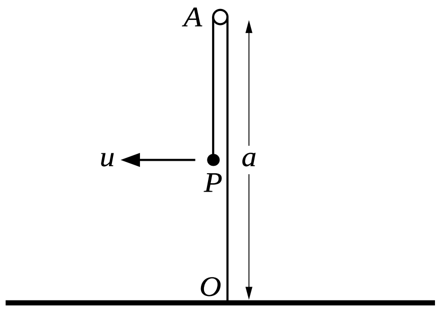
(ii) the speed of P at the instant when the string becomes slack. [3]

41. [9231/s15/21/q3]

A particle $\boldsymbol{P}$, of mass $\boldsymbol{m}$, is placed at the highest point of a fixed solid smooth sphere with centre $\boldsymbol{O}$ and radius $\boldsymbol{a}$. The particle $\boldsymbol{P}$ is given a horizontal speed $\boldsymbol{u}$ and it moves in part of a vertical circle, with centre $\boldsymbol{O}$, on the surface of the sphere. When $\boldsymbol{OP}$ makes an angle θ with the upward vertical, and $\boldsymbol{P}$ is still in contact with the surface of the sphere, the speed of $\boldsymbol{P}$ is $\boldsymbol{v}$ and the reaction of the sphere on $\boldsymbol{P}$ has magnitude $\boldsymbol{R}$. Show that $\boldsymbol{R} = \boldsymbol{mg}(3\cos\theta - 2) - \dfrac{\boldsymbol{mu}^2}{\boldsymbol{a}}$. [5]

The particle loses contact with the sphere at the instant when $\boldsymbol{v} = 2\boldsymbol{u}$. Find $\boldsymbol{u}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{g}$. [4]

42. [9231/s15/23/q10e]



One end of a light inextensible string of length $\frac{3}{2}a$ is attached to a fixed point O on a horizontal surface. The other end of the string is attached to a particle P of mass m. The string passes over a small fixed smooth peg A which is at a distance a vertically above O. The system is in equilibrium with P hanging vertically below A and the string taut. The particle is projected horizontally with speed u (see diagram). When P is at the same horizontal level as A, the tension in the string is T. Show that $T = \frac{2m}{a}(u^2 - ag)$. [3]

The ratio of the tensions in the string immediately before, and immediately after, the string loses contact with the peg is 5 : 1.

(i) Show that $u^2 = 5ag$. [6]

(ii) Find, in terms of m and g, the tension in the string when P is next at the same horizontal level as A. [5]

43. [9231/w15/21/q4]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When P is hanging at rest vertically below O, it is projected horizontally. In the subsequent motion P completes a vertical circle. The speed of P when it is at its highest point is u. Show that the least possible value of u is $\sqrt{(ag)}$. [2]

It is now given that $u = \sqrt{(ag)}$. When P passes through the lowest point of its path, it collides with, and coalesces with, a stationary particle of mass $\frac{1}{4}m$. Find the speed of the combined particle immediately after the collision. [4]

In the subsequent motion, when OP makes an angle θ with the upward vertical the tension in the string is T. Find an expression for T in terms of m, g and θ. [5]

Find the value of $\cos\theta$ when the string becomes slack. [2]

Chapter 5

Hooke's law

1. [9231/s25/31/q5]

One end of a light elastic string of natural length 0.5 m and modulus of elasticity 14 N is attached to a fixed point A on a smooth plane. The plane makes an angle α to the horizontal, where $\tan\alpha = \frac{7}{24}$. A particle P of mass 2 kg is attached to the other end of the string. The string lies along a line of greatest slope of the plane. The particle P is initially held on the plane above the level of A, where $AP = 0.8$ m. The particle P is then released from rest.

Find the maximum velocity of P during the subsequent motion. [6]

2. [9231/s25/33/q2]

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity mg. The other end of the string is attached to a fixed point O on a rough plane inclined at an angle of 30° to the horizontal. The particle P is held at rest at point O before being released. The frictional force acting on P as it slides down the plane is $\frac{11}{30}mg$.

(a) Find, in terms of a, the distance that P moves down the plane before coming to rest. [5]

(b) It is given that P remains at rest in this new position.

Find, in terms of m and g, the magnitude of the frictional force in this position. [3]

3. [9231/s25/34/q1]

A light spring of natural length a and modulus of elasticity $20mg$ is placed so that it stands vertically on a horizontal plane. The lower end of the spring is fixed to the plane. A particle of mass m is attached to the upper end of the spring.

The particle is pushed vertically downwards until the length of the spring is $\frac{3}{5}a$. The system is then released from rest.

Find the maximum extension of the spring in the subsequent motion. [5]

4. [9231/w25/31/q4]

One end of a light elastic string of natural length a and modulus of elasticity $5mg$ is attached to a fixed point O. Two particles, P and Q, of masses m and $4m$ respectively are attached to the other end of the string and they hang vertically in equilibrium. Particle Q is then detached from the string, hence releasing particle P from rest.

Find, in terms of a, the length of the string when the speed of particle P is first equal to $\sqrt{\frac{7}{5}ag}$. [6]

5. [9231/w25/32/q2]

One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to a fixed point A on a rough horizontal surface. The other end of the string is attached to a particle P of mass m. The particle and string rest on the surface. The coefficient of friction between P and the surface is μ. The particle P is initially held in equilibrium at a distance $\frac{4}{3}a$ from A. The particle is then released from rest.

(a) Given that the string never becomes slack, find the minimum value of μ. [3]

It is now given that $\mu = \frac{1}{2}$.

(b) Find the extension of the string when the particle comes to rest. [3]

6. [9231/w25/34/q6]

A and B are two fixed points at a distance $22a$ apart, with B vertically below A. A light elastic string of natural length $4a$ and modulus of elasticity $5mg$ has one end attached to A and the other end attached to a particle P of mass km. Another light elastic string of natural length $8a$ and modulus of elasticity $6mg$ has one end attached to B and the other end attached to P. Particle P is vertically above B.

(a) Show that, when the system is in equilibrium, $BP = \frac{57a - 2ak}{4}$. [5]

The particle P is pulled vertically upwards so that $BP = 18a$, and is then released from rest. In its subsequent motion, P first comes to instantaneous rest at the point where $BP = 8a$.

(b) Find the value of k. [4]

7. [9231/s24/31/q2]

The points A and B are at the same horizontal level a distance $4a$ apart. The ends of a light elastic string, of natural length $4a$ and modulus of elasticity λ, are attached to A and B. A particle P of mass m is attached to the midpoint of the string. The system is in equilibrium with P at a distance $\frac{3}{2}a$ below M, the midpoint of AB.

(a) Find λ in terms of m and g. [3]

The particle P is pulled down vertically and released from rest at a distance $\frac{8}{3}a$ below M.

(b) Find, in terms of a and g, the speed of P as it passes through M in the subsequent motion. [4]

8. [9231/s24/33/q2]

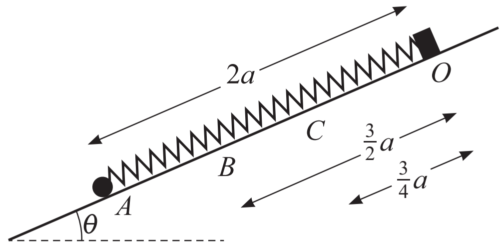
A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $2mg$. A particle Q of mass km is attached to the other end of the string. Particle P lies on a smooth horizontal table. The string has part of its length in contact with the table and then passes through a small smooth hole H in the table.

Particle P moves in a horizontal circle on the surface of the table with constant speed $\sqrt{\frac{1}{2}ga}$. Particle Q hangs in equilibrium vertically below the hole with $HQ = \frac{1}{4}a$.

(a) Find, in terms of a, the extension in the string. [4]

(b) Find the value of k. [2]

9. [9231/s24/33/q4]



A light spring of natural length a and modulus of elasticity kmg is attached to a fixed point O on a smooth plane inclined to the horizontal at an angle θ, where $\sin\theta = \frac{3}{4}$. A particle of mass m is attached to the lower end of the spring and is held at the point A on the plane, where $OA = 2a$ and OA is along a line of greatest slope of the plane (see diagram).

The particle is released from rest and is moving with speed V when it passes through the point B on the plane, where $OB = \frac{3}{2}a$. The speed of the particle is $\frac{1}{2}V$ when it passes through the point C on the plane, where $OC = \frac{3}{4}a$.

Find the value of k. [7]

10. [9231/w24/31/q3]

A particle P of mass m kg is attached to one end of a light elastic string of natural length 2 m and modulus of elasticity $2mg$ N. The other end of the string is attached to a fixed point O. The particle P hangs in equilibrium vertically below O. The particle P is pulled down vertically a distance d m below its equilibrium position and released from rest.

(a) Given that the particle just reaches O in the subsequent motion, find the value of d. [6]

(b) Hence find the speed of P when it is 2 m below O. [2]

11. [9231/w24/32/q1]

A particle of mass 2 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 100 N. The other end of the string is attached to a fixed point O on a smooth horizontal surface. The particle is moving in a horizontal circle about O with the string taut and with constant angular speed 5 radians per second.

Find the extension of the string. [3]

12. [9231/w24/32/q2]

A particle P of mass m is attached to one end of a light elastic spring of natural length a and modulus of elasticity $5mg$. The other end of the spring is attached to a fixed point O. The spring hangs vertically with P below O. The particle P is pulled down vertically and released from rest when the length of the spring is $\frac{3}{2}a$.

Find the distance of P below O when P first comes to instantaneous rest. [4]

13. [9231/s23/31/q1]

One end of a light elastic string, of natural length a and modulus of elasticity $3mg$, is attached to a fixed point O. The other end of the string is attached to a particle P of mass m. The string hangs with P vertically below O. The particle P is pulled vertically downwards so that the extension of the string is $2a$. The particle P is then released from rest.

(a) Find the speed of P when it is at a distance $\frac{3}{4}a$ below O. [3]

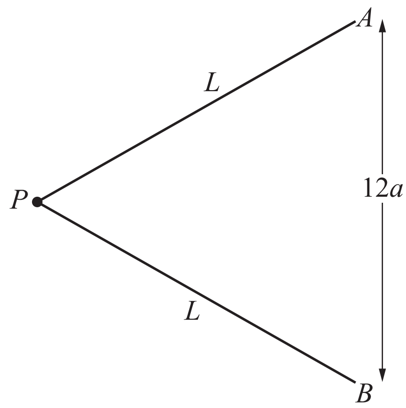
(b) Find the initial acceleration of P when it is released from rest. [2]

14. [9231/s23/33/q2]

One end of a light elastic string, of natural length a and modulus of elasticity λmg, is attached to a fixed point O. The string lies on a smooth horizontal surface. A particle P of mass m is attached to the other end of the string. The particle P is projected in the direction OP. When the length of the string is $\frac{4}{3}a$, the speed of P is $\sqrt{2ag}$. When the length of the string is $\frac{5}{3}a$, the speed of P is $\frac{1}{2}\sqrt{2ag}$.

Find the value of λ. [4]

15. [9231/w23/31/q4]

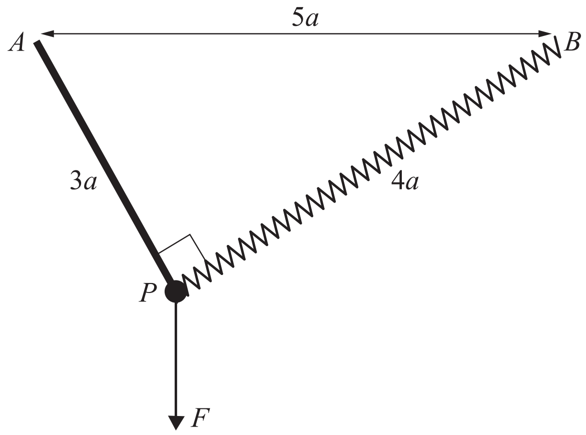


A light elastic string has natural length $8a$ and modulus of elasticity $5mg$. A particle P of mass m is attached to the midpoint of the string. The ends of the string are attached to points A and B which are a distance $12a$ apart on a smooth horizontal table. The particle P is held on the table so that $AP = BP = L$ (see diagram). The particle P is released from rest. When P is at the midpoint of AB it has speed $\sqrt{80ag}$.

(a) Find L in terms of a. [5]

(b) Find the initial acceleration of P in terms of g. [3]

16. [9231/w23/32/q7]



A particle P of mass m is attached to one end of a light rod of length $3a$. The other end of the rod is able to pivot smoothly about the fixed point A. The particle is also attached to one end of a light spring of natural length a and modulus of elasticity kmg. The other end of the spring is attached to a fixed point B. The points A and B are in a horizontal line, a distance $5a$ apart, and these two points and the rod are in a vertical plane.

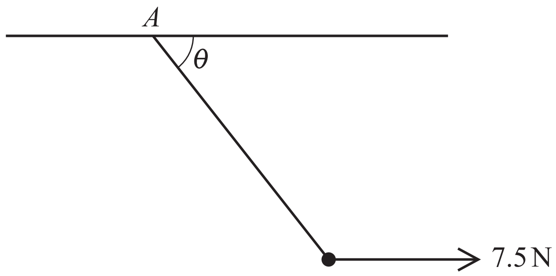
Initially, P is held in equilibrium by a vertical force F with the stretched length of the spring equal to $4a$ (see diagram). The particle is released from rest in this position and has a speed of $\frac{6}{5}\sqrt{2ag}$ when the rod becomes horizontal.

(a) Find the value of k. [5]

(b) Find F in terms of m and g. [2]

(c) Find, in terms of m and g, the tension in the rod immediately before it is released. [2]

17. [9231/s22/31/q1]



A particle of weight 10 N is attached to one end of a light elastic string. The other end of the string is attached to a fixed point A on a horizontal ceiling. A horizontal force of 7.5 N acts on the particle. In the equilibrium position, the string makes an angle θ with the ceiling (see diagram). The string has natural length 0.8 m and modulus of elasticity 50 N.

(a) Find the tension in the string. [2]

(b) Find the vertical distance between the particle and the ceiling. [3]

18. [9231/s22/33/q2]

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $\frac{4}{3}mg$. The other end of the string is attached to a fixed point O on a rough horizontal surface. The particle is at rest on the surface with the string at its natural length. The coefficient of friction between P and the surface is $\frac{1}{3}$. The particle is projected along the surface in the direction OP with a speed of $\frac{1}{2}\sqrt{ga}$.

Find the greatest extension of the string during the subsequent motion. [5]

19. [9231/w22/31/q2]

A light elastic string has natural length a and modulus of elasticity $4mg$. One end of the string is fixed to a point O on a smooth horizontal surface. A particle P of mass m is attached to the other end of the string. The particle P is projected along the surface in the direction OP. When the length of the string is $\frac{5}{4}a$, the speed of P is v. When the length of the string is $\frac{3}{2}a$, the speed of P is $\frac{1}{2}v$.

(a) Find an expression for v in terms of a and g. [4]

(b) Find, in terms of g, the acceleration of P when the stretched length of the string is $\frac{3}{2}a$. [2]

20. [9231/w22/32/q3]

One end of a light elastic string, of natural length a and modulus of elasticity $\frac{16}{3}Mg$, is attached to a fixed point O. A particle P of mass $4M$ is attached to the other end of the string and hangs vertically in equilibrium. Another particle of mass $2M$ is attached to P and the combined particle is then released from rest. The speed of the combined particle when it has descended a distance $\frac{1}{4}a$ is v.

Find an expression for v in terms of g and a. [6]

21. [9231/s21/31/q3]

One end of a light elastic string, of natural length a and modulus of elasticity kmg, is attached to a fixed point A. The other end of the string is attached to a particle P of mass $4m$. The particle P hangs in equilibrium a distance x vertically below A.

(a) Show that $k = \frac{4a}{x-a}$. [1]

An additional particle, of mass $2m$, is now attached to P and the combined particle is released from rest at the original equilibrium position of P. When the combined particle has descended a distance $\frac{1}{3}a$, its speed is $\frac{1}{3}\sqrt{ga}$.

(b) Find x in terms of a. [6]

22. [9231/s21/33/q2]

One end of a light elastic string of natural length 0.8 m and modulus of elasticity 36 N is attached to a fixed point O on a smooth plane. The plane is inclined at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. A particle P of mass 2 kg is attached to the other end of the string. The string lies along a line of greatest slope of the plane with the particle below the level of O. The particle is projected with speed $\sqrt{2}\,\text{m s}^{-1}$ directly down the plane from the position where OP is equal to the natural length of the string.

Find the maximum extension of the string during the subsequent motion. [5]

23. [9231/w21/31/q1]

One end of a light elastic string, of natural length a and modulus of elasticity $3mg$, is attached to a fixed point O on a smooth horizontal plane. A particle P of mass m is attached to the other end of the string and moves in a horizontal circle with centre O. The speed of P is $\sqrt{\frac{4}{3}ga}$.

Find the extension of the string. [4]

24. [9231/w21/31/q3]

A light elastic string has natural length a and modulus of elasticity $12mg$. One end of the string is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The particle hangs in equilibrium vertically below O. The particle is pulled vertically down and released from rest with the extension of the string equal to e, where $e > \frac{1}{3}a$. In the subsequent motion the particle has speed $\sqrt{2ga}$ when it has ascended a distance $\frac{1}{3}a$.

Find e in terms of a. [6]

25. [9231/w21/32/q2]

A light spring AB has natural length a and modulus of elasticity $5mg$. The end A of the spring is attached to a fixed point on a smooth horizontal surface. A particle P of mass m is attached to the end B of the spring. The spring and particle P are at rest on the surface.

Another particle Q of mass km is moving with speed $\sqrt{4ga}$ along the horizontal surface towards P in the direction BA. The particles P and Q collide directly and coalesce. In the subsequent motion the greatest amount by which the spring is compressed is $\frac{1}{5}a$.

Find the value of k. [6]

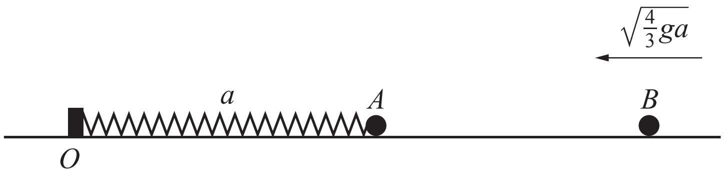
26. [9231/s20/31/q3]

One end of a light elastic spring, of natural length a and modulus of elasticity $5mg$, is attached to a fixed point A. The other end of the spring is attached to a particle P of mass m. The spring hangs with P vertically below A. The particle P is released from rest in the position where the extension of the spring is $\frac{1}{2}a$.

(a) Show that the initial acceleration of P is $\frac{3}{2}g$ upwards. [3]

(b) Find the speed of P when the spring first returns to its natural length. [4]

27. [9231/s20/33/q7]



One end of a light spring of natural length a and modulus of elasticity $4mg$ is attached to a fixed point O. The other end of the spring is attached to a particle A of mass km, where k is a constant. Initially the spring lies at rest on a smooth horizontal surface and has length a. A second particle B, of mass m, is moving towards A with speed $\sqrt{\frac{4}{3}ga}$ along the line of the spring from the opposite direction to O (see diagram).

The particles A and B collide and coalesce. At a point C in the subsequent motion, the length of the spring is $\frac{3}{4}a$ and the speed of the combined particle is half of its initial speed.

(a) Find the value of k. [6]

At the point C the horizontal surface becomes rough, with coefficient of friction μ between the combined particle and the surface. The deceleration of the combined particle at C is $\frac{9}{20}g$.

(b) Find the value of μ. [4]

28. [9231/w20/31/q1]

A particle P of mass m is placed on a fixed smooth plane which is inclined at an angle θ to the horizontal. A light spring, of natural length a and modulus of elasticity $3mg$, has one end attached to P and the other end attached to a fixed point O at the top of the plane. The spring lies along a line of greatest slope of the plane. The system is released from rest with the spring at its natural length.

Find, in terms of a and θ, an expression for the greatest extension of the spring in the subsequent motion. [3]

29. [9231/w20/31/q3]

One end of a light elastic string, of natural length a and modulus of elasticity $4mg$, is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The particle moves in a horizontal circle with a constant angular speed $\sqrt{\frac{g}{a}}$ with the string inclined at an angle θ to the downward vertical through O. The length of the string during this motion is $(k+1)a$.

(a) Find the value of k. [4]

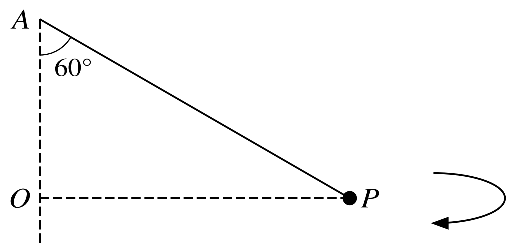
(b) Find the value of $\cos\theta$. [2]

30. [9231/w20/32/q6]

One end of a light elastic string, of natural length a and modulus of elasticity k, is attached to a particle P of mass m. The other end of the string is attached to a fixed point Q. The particle P is projected vertically upwards from Q. When P is moving upwards and at a distance $\frac{4}{3}a$ directly above Q, it has a speed $\sqrt{2ga}$. At this point, its acceleration is $\frac{7}{3}g$ downwards.

Show that $k = 4mg$ and find in terms of a the greatest height above Q reached by P. [8]

31. [9709/m19/52/q4]



A particle P of mass 0.3 kg is attached to a fixed point A by a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The particle P moves in a horizontal circle which has centre O. It is given that AO is vertical and that angle OAP is 60° (see diagram). Calculate the speed of P. [6]

32. [9709/m19/52/q5]

A particle P of mass 0.3 kg is attached to one end of a light elastic string of natural length 0.6 m and modulus of elasticity 24 N. The other end of the string is attached to a fixed point O. The particle P is released from rest at the point 0.4 m vertically below O.

(i) Find the greatest speed of P. [5]

(ii) Calculate the greatest distance of P below O. [3]

33. [9709/s19/51/q5]

A particle P of mass 0.4 kg is attached to one end of a light elastic string of natural length 0.5 m and modulus of elasticity 6 N. The other end of the string is attached to a fixed point O. The particle P is released from rest at the point $(0.5 + x)$ m vertically below O. The particle P comes to instantaneous rest at O.

(i) Find x. [3]

(ii) Find the greatest speed of P. [5]

34. [9709/s19/52/q5]

A light elastic string has natural length a m and modulus of elasticity λ N. When the length of the string is 1.6 m the tension is 4 N. When the length of the string is 2 m the tension is 6 N.

(i) Find the values of a and λ. [5]

One end of the string is attached to a fixed point O on a smooth horizontal surface. The other end of the string is attached to a particle P of mass 0.2 kg. The particle P moves with constant speed on the surface in a circle with centre O and radius 1.9 m.

(ii) Find the speed of P. [3]

35. [9709/w19/51/q5]

A particle P of mass 0.3 kg is attached to one end of a light elastic string of natural length 0.6 m and modulus of elasticity 9 N. The other end of the string is attached to a fixed point O on a smooth plane inclined at 30° to the horizontal. OA is a line of greatest slope of the plane with A below the level of O and $OA = 0.8$ m. The particle P is released from rest at A.

(i) Find the initial acceleration of P. [4]

(ii) Find the greatest speed of P. [5]

36. [9709/w19/52/q1]

A particle of mass 0.3 kg is attached to one end of a light elastic string of natural length 0.6 m and modulus of elasticity 9 N. The other end of the string is attached to a fixed point O on a smooth horizontal surface. The particle is projected horizontally from O with speed 4 m s^{-1}. Find the greatest distance of the particle from O. [3]

37. [9709/w19/52/q3]

A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 0.6 m and modulus of elasticity 12 N. The other end of the string is attached to a fixed point O. The particle P is projected vertically downwards with speed $2\,\mathrm{m\,s^{-1}}$ from the point 0.5 m vertically below O. For an instant when the acceleration of P is $4\,\mathrm{m\,s^{-2}}$ downwards, find the extension of the string and the speed of P. [6]

38. [9231/w18/21/q4]

A uniform rod AB of length $4a$ and weight W is smoothly hinged to a vertical wall at the end A. The rod is held at an angle θ above the horizontal by a light elastic string. One end of the string is attached to the point C on the rod, where $AC = 3a$. The other end of the string is attached to a point D on the wall, with D vertically above A and such that angle $ACD = 2\theta$. A particle of weight $\frac{1}{2}W$ is attached to the rod at B. It is given that $\tan\theta = \frac{8}{15}$.

(i) Show that the tension in the string is $\frac{17}{12}W$. [4]

(ii) Find the magnitude and direction of the reaction at the hinge. [5]

(iii) Given that the natural length of the string is $2a$, find its modulus of elasticity. [2]

39. [9709/m18/52/q3]

A small ball B is connected to one end of a light elastic string of natural length 0.4 m and modulus of elasticity 12 N. The other end of the string is attached to a fixed point A. The ball is projected with speed 1 m s^{-1} vertically downwards from a position 0.4 m vertically below A, and reaches its greatest speed at the point 0.7 m below A.

(i) Show that the mass of B is 0.9 kg. [2]

(ii) Calculate the greatest speed of B. [4]

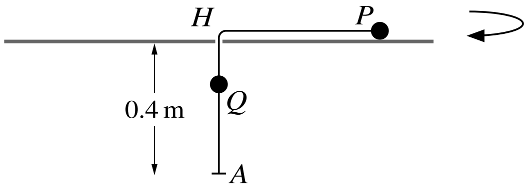
40. [9709/s18/51/q5]

A particle P of mass 0.7 kg is attached by a light elastic string to a fixed point O on a smooth plane inclined at an angle of 30° to the horizontal. The natural length of the string is 0.5 m and the modulus of elasticity is 20 N. The particle P is projected up the line of greatest slope through O from a point A below the level of O. The initial kinetic energy of P is 1.8 J and the initial elastic potential energy in the string is also 1.8 J.

(i) Find the distance OA. [2]

(ii) Find the greatest speed of P in the motion. [6]

41. [9709/s18/51/q6]



A particle P of mass 0.2 kg is attached to one end of a light inextensible string of length 0.6 m. The other end of the string is attached to a particle Q of mass 0.3 kg. The string passes through a small hole H in a smooth horizontal surface. A light elastic string of natural length 0.3 m and modulus of elasticity 15 N joins Q to a fixed point A which is 0.4 m vertically below H. The particle P moves on the surface in a horizontal circle with centre H (see diagram).

(i) Calculate the greatest possible speed of P for which the elastic string is not extended. [4]

(ii) Find the distance HP given that the angular speed of P is $8\,\text{rad}\,\text{s}^{-1}$. [5]

42. [9709/s18/52/q2]

One end of a light elastic string is attached to a fixed point O. The other end of the string is attached to a particle P of mass 0.4 kg. The string has natural length 0.6 m and modulus of elasticity 24 N. The particle is released from rest at O. Find the two possible values of the distance OP for which the particle has speed $1.5\,\text{m}\,\text{s}^{-1}$. [6]

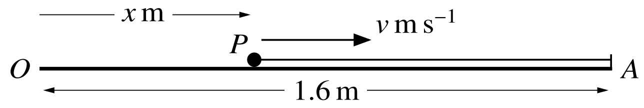
43. [9709/w18/51/q3]

A particle P of mass 0.4 kg is attached to a fixed point O by a light elastic string of natural length 0.5 m and modulus of elasticity 20 N. The particle P is released from rest at O.

(i) Find the greatest speed of P in the subsequent motion. [4]

(ii) Find the distance below O of the point at which P comes to instantaneous rest. [3]

44. [9709/w18/51/q4]



A particle P of mass 0.5 kg is projected along a smooth horizontal surface towards a fixed point A. Initially P is at a point O on the surface, and after projection, P has a displacement from O of x m and velocity v m s^{-1}. The particle P is connected to A by a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The distance OA is 1.6 m (see diagram). The motion of P is resisted by a force of magnitude $24x^2$ N.

(i) Show that $v\dfrac{dv}{dx} = 32 - 40x - 48x^2$ while P is in motion and the string is stretched. [3]

The maximum value of v is 4.5.

(ii) Find the initial value of v. [5]

45. [9709/w18/52/q3]

A particle P of mass 0.4 kg is projected horizontally along a smooth horizontal plane from a point O. After projection the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from O is x m. A force of magnitude $8x$ N directed away from O acts on P and a force of magnitude $(2e^{-x} + 4)$ N opposes the motion of P. One end of a light elastic string of natural length 0.5 m is attached to O and the other end of the string is attached to P.

(i) Show that $v\dfrac{dv}{dx} = 20x - 10 - 5e^{-x}$ before the elastic string becomes taut. [2]

(ii) Given that the initial velocity of P is $6\,\mathrm{m\,s^{-1}}$, find v when the string first becomes taut. [3]

When the string is taut, the acceleration of P is proportional to e^{-x}.

(iii) Find the modulus of elasticity of the string. [2]

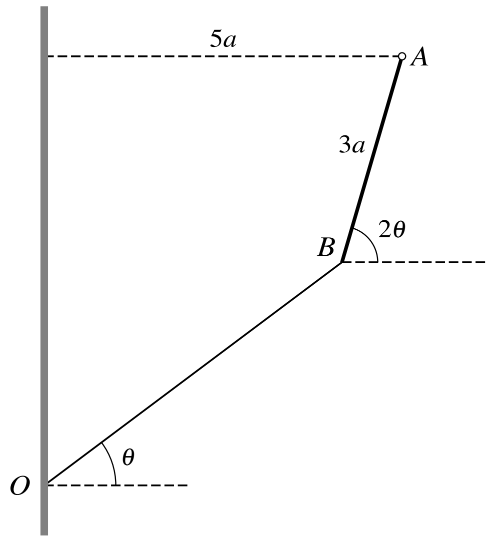
46. [9709/w18/52/q5]

A particle P of mass 0.7 kg is attached to a fixed point O by a light elastic string of natural length 0.6 m and modulus of elasticity 15 N. The particle P is projected vertically downwards from the point A, 0.8 m vertically below O. The initial speed of P is $2\,\mathrm{m\,s^{-1}}$.

(i) Find the distance below A of the point at which P comes to instantaneous rest. [4]

(ii) Find the greatest speed of P in the motion. [4]

47. [9231/s17/23/q4]



A uniform rod AB of length $3a$ and weight W is freely hinged to a fixed point at the end A. The end B is below the level of A and is attached to one end of a light elastic string of natural length $4a$. The other end of the string is attached to a point O on a vertical wall. The horizontal distance between A and the wall is $5a$. The string and the rod make angles θ and 2θ respectively with the horizontal (see diagram). The system is in equilibrium with the rod and the string in the same vertical plane. It is given that $\sin\theta = \frac{3}{5}$ and you may use the fact that $\cos 2\theta = \frac{7}{25}$.

(i) Find the tension in the string in terms of W. [3]

(ii) Find the modulus of elasticity of the string in terms of W. [4]

(iii) Find the angle that the force acting on the rod at A makes with the horizontal. [3]

48. [9709/m17/52/q7]

One end of a light elastic string of natural length 0.6 m and modulus of elasticity 24 N is attached to a fixed point O. The other end of the string is attached to a particle P of mass 0.4 kg which hangs in equilibrium vertically below O.

(i) Calculate the extension of the string. [2]

P is projected vertically downwards from the equilibrium position with speed $5\,\mathrm{m\,s^{-1}}$.

(ii) Calculate the distance P travels before it is first at instantaneous rest. [4]

When P is first at instantaneous rest a stationary particle of mass 0.4 kg becomes attached to P.

(iii) Find the greatest speed of the combined particle in the subsequent motion. [4]

49. [9709/s17/51/q2]

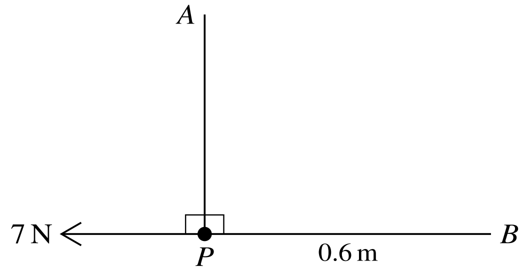


Fig. 1

One end of a light inextensible string is attached to a fixed point A. The other end of the string is attached to a particle P of mass m kg which hangs vertically below A. The particle is also attached to one end of a light elastic string of natural length 0.25 m. The other end of this string is attached to a point B which is 0.6 m from P and on the same horizontal level as P. Equilibrium is maintained by a horizontal force of magnitude 7 N applied to P (see Fig. 1).

(i) Calculate the modulus of elasticity of the elastic string. [2]

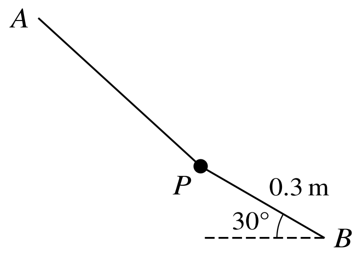


Fig. 2

P is released from rest by removing the 7 N force. In its subsequent motion P first comes to instantaneous rest at a point where $BP = 0.3$ m and the elastic string makes an angle of 30° with the horizontal (see Fig. 2).

(ii) Find the value of m. [4]

50. [9709/s17/51/q6]

A particle P of mass 0.15 kg is attached to one end of a light elastic string of natural length 0.4 m and modulus of elasticity 12 N. The other end of the string is attached to a fixed point A. The particle P moves in a horizontal circle which has its centre vertically below A, with the string inclined at $\theta°$ to the vertical and $AP = 0.5$ m.

(i) Find the angular speed of P and the value of θ. [5]

(ii) Calculate the difference between the elastic potential energy stored in the string and the kinetic energy of P. [4]

51. [9709/s17/52/q1]

A particle P of mass 0.2 kg moves with speed $4\,\mathrm{m\,s^{-1}}$ and angular speed $5\,\mathrm{rad\,s^{-1}}$ in a horizontal circle on a smooth surface. P is attached to one end of a light elastic string of natural length 0.6 m. The other end of the string is attached to the point on the surface which is the centre of the circular motion of P.

(i) Find the radius of this circle. [1]

(ii) Find the modulus of elasticity of the string. [4]

52. [9709/s17/52/q5]

A particle of mass 0.3 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 6 N. The other end of the string is attached to a fixed point O. The particle is projected vertically downwards from O with initial speed $2\ \mathrm{m\,s^{-1}}$.

(i) Calculate the greatest speed of the particle during its descent. [5]

(ii) Find the greatest distance of the particle below O. [3]

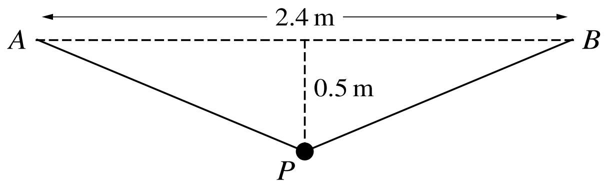
53. [9709/w17/51/q5]

One end of a light elastic string of natural length 0.8 m and modulus of elasticity 24 N is attached to a fixed point O. The other end of the string is attached to a particle P of mass 0.3 kg. P is projected vertically upwards with speed 4 m s^{-1} from a position 1.2 m vertically below O.

(i) Calculate the speed of the particle at the position where it is moving with zero acceleration. [5]

(ii) Show that the particle moves 1.2 m while moving upwards with constant deceleration. [3]

54. [9709/w17/52/q4]



A light elastic string has natural length 2 m and modulus of elasticity 39 N. The ends of the string are attached to fixed points A and B which are at the same horizontal level and 2.4 m apart. A particle P of mass m kg is attached to the mid-point of the string and hangs in equilibrium at a point 0.5 m below AB (see diagram).

(i) Show that $m = 0.9$. [4]

P is projected vertically downwards from the equilibrium position, and comes to instantaneous rest at a point 1.6 m below AB.

(ii) Calculate the speed of projection of P. [5]

55. [9709/w17/52/q6]

One end of a light elastic string of natural length 0.4 m and modulus of elasticity 8 N is attached to a fixed point O on a smooth horizontal plane. The other end of the string is attached to a particle P of mass 0.2 kg which moves on the plane in a circular path with centre O. The speed of P is $v\,\text{m}\,\text{s}^{-1}$ and the extension of the string is x m.

(i) Given that $v = 2.5$, find x. [4]

It is given instead that the kinetic energy of P is twice the elastic potential energy stored in the string.

(ii) Form two simultaneous equations and hence find x and v. [5]

56. [9709/m16/52/q5]

A particle P of mass 0.6 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 24 N. The other end of the string is attached to a fixed point A, and P hangs in equilibrium.

(i) Calculate the extension of the string. [2]

P is projected vertically downwards from the equilibrium position with speed 4.5 m s^{-1}.

(ii) Find the distance AP when the speed of P is 3.5 m s^{-1} and P is below the equilibrium position. [4]

(iii) Calculate the speed of P when it is 0.5 m above the equilibrium position. [3]

57. [9709/s16/51/q7]

A particle P is attached to one end of a light elastic string of natural length 1.2 m and modulus of elasticity 12 N. The other end of the string is attached to a fixed point O on a smooth plane inclined at an angle of 30° to the horizontal. P rests in equilibrium on the plane, 1.6 m from O.

(i) Calculate the mass of P. [2]

A particle Q, with mass equal to the mass of P, is projected up the plane along a line of greatest slope. When Q strikes P the two particles coalesce. The combined particle remains attached to the string and moves up the plane, coming to instantaneous rest after moving 0.2 m.

(ii) Show that the initial kinetic energy of the combined particle is 1 J. [4]

The combined particle subsequently moves down the plane.

(iii) Calculate the greatest speed of the combined particle in the subsequent motion. [5]

58. [9709/s16/52/q2]

One end of a light elastic string of natural length 0.4 m is attached to a fixed point O. The other end of the string is attached to a particle of weight 5 N which hangs in equilibrium 0.6 m vertically below O.

(i) Find the modulus of elasticity of the string. [2]

The particle is projected vertically upwards from the equilibrium position and comes to instantaneous rest after travelling 0.3 m upwards.

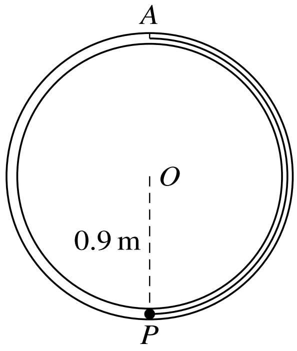
(ii) Calculate the speed of projection of the particle. [3]

(iii) Calculate the greatest extension of the string in the subsequent motion. [3]

59. [9709/w16/51/q1]

A particle P of mass 0.3 kg moves in a circle with centre O on a smooth horizontal surface. P is attached to O by a light elastic string of modulus of elasticity 12 N and natural length l m. The speed of P is $4\,\mathrm{m\,s^{-1}}$, and the radius of the circle in which it moves is $2l$ m. Calculate l. [4]

60. [9709/w16/51/q6]



The diagram shows a smooth narrow tube formed into a fixed vertical circle with centre O and radius 0.9 m. A light elastic string with modulus of elasticity 8 N and natural length 1.2 m has one end attached to the highest point A on the inside of the tube. The other end of the string is attached to a particle P of mass 0.2 kg. The particle is released from rest at the lowest point on the inside of the tube. By considering energy, calculate

(i) the speed of P when it is at the same horizontal level as O, [4]

(ii) the speed of P at the instant when the string becomes slack. [3]

61. [9709/w16/52/q2]

A particle P of mass 0.5 kg is attached to one end of a light elastic string with modulus of elasticity 24 N and natural length 0.6 m. The other end of the string is attached to a fixed point A. The particle P hangs in equilibrium vertically below A.

(i) Find the distance AP. [2]

The particle P is raised to A and released from rest.

(ii) Calculate the greatest speed of P in the subsequent motion. [3]

62. [9709/s15/51/q1]

One end of a light elastic string of natural length 0.7 m is attached to a fixed point A on a smooth horizontal surface. The other end of the string is attached to a particle P of mass 0.3 kg which is held at a point B on the horizontal surface, where $AB = 1.2$ m. It is given that P is released from rest at B and that when $AP = 0.9$ m, the particle has speed $4\,\text{m}\,\text{s}^{-1}$. Calculate the modulus of elasticity of the string. [3]

63. [9709/s15/51/q5]

A particle P of mass 0.3 kg is attached to one end of a light elastic string of natural length 0.9 m and modulus of elasticity 18 N. The other end of the string is attached to a fixed point O which is 3 m above the ground.

(i) Find the extension of the string when P is in the equilibrium position. [2]

P is projected vertically downwards from the equilibrium position with initial speed $6\,\mathrm{m\,s^{-1}}$. At the instant when the tension in the string is 12 N the string breaks. P continues to descend vertically.

(ii) (a) Calculate the height of P above the ground at the instant when the string breaks. [2]

(b) Find the speed of P immediately before it strikes the ground. [4]

64. [9709/s15/52/q2]

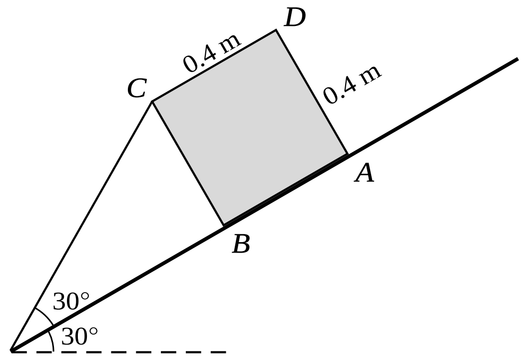
One end of a light elastic string of natural length 0.5 m and modulus of elasticity 30 N is attached to a fixed point O. The other end of the string is attached to a particle P which hangs in equilibrium vertically below O, with $OP = 0.8$ m.

(i) Show that the mass of P is 1.8 kg. [2]

The particle is pulled vertically downwards and released from rest from the point where $OP = 1.2$ m.

(ii) Find the speed of P at the instant when the string first becomes slack. [3]

65. [9709/s15/52/q5]



A uniform solid cube with edges of length 0.4 m rests in equilibrium on a rough plane inclined at an angle of 30° to the horizontal. $ABCD$ is a cross-section through the centre of mass of the cube, with AB along a line of greatest slope. B lies below the level of A. One end of a light elastic string with modulus of elasticity 12 N and natural length 0.4 m is attached to C. The other end of the string is attached to a point below the level of B on the same line of greatest slope, such that the string makes an angle of 30° with the plane (see diagram). The cube is on the point of toppling. Find

(i) the tension in the string, [3]

(ii) the weight of the cube. [4]

66. [9709/s15/53/q3]

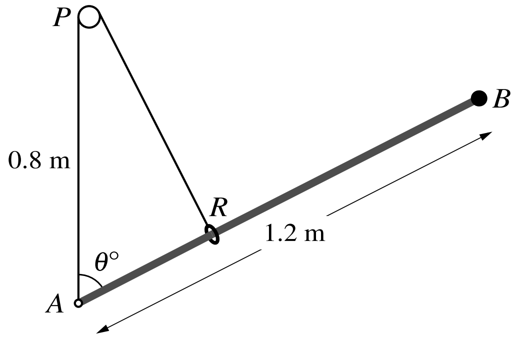
One end of a light elastic string of natural length 0.4 m and modulus of elasticity 20 N is attached to a fixed point A on a smooth plane inclined at 30° to the horizontal. The other end of the string is attached to a particle P of mass 0.5 kg which rests in equilibrium on the plane.

(i) Calculate the extension of the string. [2]

P is projected down the plane from the equilibrium position with speed 5 m s^{-1}. The extension of the string is e m when the speed of the particle is 2 m s^{-1} for the first time.

(ii) Find e. [4]

67. [9709/w15/51/q2]



A uniform rigid rod AB of length 1.2 m and weight 8 N has a particle of weight 2 N attached at the end B. The end A of the rod is freely hinged to a fixed point. One end of a light elastic string of natural length 0.8 m and modulus of elasticity 20 N is attached to the hinge. The string passes over a small smooth pulley P fixed 0.8 m vertically above the hinge. The other end of the string is attached to a small light smooth ring R which can slide on the rod. The system is in equilibrium with the rod inclined at an angle $\theta°$ to the vertical (see diagram).

(i) Show that the tension in the string is $20 \sin \theta$ N. [1]

(ii) Explain why the part of the string attached to the ring is perpendicular to the rod. [1]

(iii) Find θ. [3]

68. [9709/w15/51/q5]

A particle P of mass 0.2 kg is attached to one end of a light elastic string of natural length 0.75 m and modulus of elasticity 21 N. The other end of the string is attached to a fixed point A which is 0.8 m vertically above a smooth horizontal surface. P rests in equilibrium on the surface.

(i) Find the magnitude of the force exerted on P by the surface. [2]

P is now projected horizontally along the surface with speed $3\,\mathrm{m\,s^{-1}}$.

(ii) Calculate the extension of the string at the instant when P leaves the surface. [3]

(iii) Hence find the speed of P at the instant when it leaves the surface. [3]

69. [9709/w15/53/q7]

A particle P of mass M kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 12.5 N. The other end of the string is attached to a fixed point A. The particle is released from rest at A and falls vertically until it comes to instantaneous rest at the point B. The greatest speed of P during its descent is $4.4\,\mathrm{m\,s^{-1}}$ when the extension of the string is e m.

(i) Show that $e = 0.64M$. [2]

(ii) Find a second equation in e and M, and hence find M. [6]

(iii) Calculate the distance AB. [3]

Chapter 6

Linear motion under a variable force

1. [9231/s25/31/q1]

A particle P of mass 8 kg is moving in a straight horizontal line. At time t s, P has displacement x m from a fixed point O on the line and velocity v m s^{-1}. The only horizontal force acting on P has magnitude $(x^3 + 4x)$ N and acts in the direction OP. When $t = 0$, $x = 0$ and $v = 1$.

(a) Find an expression for v in terms of x, giving your answer in the form $v = ax^2 + b$, where a and b are constants to be determined. [3]

(b) Find an expression for x in terms of t. [2]

2. [9231/s25/33/q3]

A ball of mass m kg is projected vertically upwards with initial speed U m s^{-1} and moves under gravity. At time t s after projection, the ball has travelled a distance x m and its speed is v m s^{-1}. There is a resistive force of magnitude mkv^2 N, where k is a positive constant.

(a) Show that the distance travelled by the ball when it is moving upwards is $x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right)$. [4]

It is given that $k = 0.025$ and that $U = 20$.

(b) Find the time taken for the ball to reach its maximum height. [4]

3. [9231/s25/34/q2]

A particle P of mass m kg moves along a horizontal straight line against a resistive force of magnitude $2mv^3$ N, where v m s^{-1} is the velocity of P at time t s. When $t = 0$, $v = 1$.

(a) Find an expression for v in terms of t. [4]

(b) Find the displacement of P from its initial position when $t = 6$. [3]

4. [9231/w25/31/q3]

A particle P is moving in a straight horizontal line. At time t s, the displacement of P from a fixed point O on the line is x m and the velocity of P is $v\,\mathrm{m\,s^{-1}}$. The acceleration of P is $\frac{1}{2}(v^2+4)\,\mathrm{m\,s^{-2}}$ in the direction PO. Initially P is at O and is moving with velocity $2\,\mathrm{m\,s^{-1}}$.

(a) Find an expression for x in terms of t. [5]

(b) Find the time when P next goes through O. [2]

5. [9231/w25/32/q7]

A particle P of mass m kg moving along a rough horizontal table has displacement x m from a fixed point O on the table and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The particle P is subject to a resistive force of magnitude $mgkv$ N, where k is a positive constant, and a frictional force of magnitude μmg. The particle P is initially at O with speed $U\,\mathrm{m\,s^{-1}}$.

(a) Show that $t = \dfrac{1}{gk}\ln\left(\dfrac{kU+\mu}{kv+\mu}\right)$. [4]

....

It is given that $U = 10$, $k = 0.04$ and $\mu = 0.2$.

(b) Find the distance P moves before coming to rest. [4]

(c) Find the average speed of P over the period it is moving. [2]

6. [9231/w25/34/q5]

A particle P of mass m kg is projected vertically upwards from a point O with an initial speed of $20\,\mathrm{m\,s^{-1}}$ and moves under gravity. There is a resistive force of magnitude $0.025mv^2$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s after projection. The displacement of P from O is x m at time t s after projection.

(a) Find an expression for v in terms of x, while P is moving upwards. [5]

(b) Find an expression for v in terms of t, while P is moving upwards. [4]

7. [9231/s24/31/q6]

A particle P of mass 2 kg moving on a horizontal straight line has displacement x m from a fixed point O on the line and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The only horizontal force acting on P has magnitude $\frac{1}{10}(2v-1)^2\mathrm{e}^{-t}$ N and acts towards O. When $t = 0$, $x = 1$ and $v = 3$.

(a) Find an expression for v in terms of t. [5]

(b) Find an expression for x in terms of t. [4]

8. [9231/s24/33/q7]

A parachutist of mass m kg opens his parachute when he is moving vertically downwards with a speed of $50\,\mathrm{m\,s^{-1}}$. At time t s after opening his parachute, he has fallen a distance x m from the point where he opened his parachute, and his speed is $v\,\mathrm{m\,s^{-1}}$. The forces acting on him are his weight and a resistive force of magnitude mv N.

(a) Find an expression for v in terms of t. [6]

(b) Find an expression for x in terms of t. [3]

(c) Find the distance that the parachutist has fallen, since opening his parachute, when his speed is $15\,\mathrm{m\,s^{-1}}$. [2]

9. [9231/w24/31/q5]

A particle P of mass 2 kg moving on a horizontal straight line has displacement x m from a fixed point O on the line and velocity v m s^{-1} at time t s. The only horizontal force acting on P is a variable force F N which can be expressed as a function of t. It is given that

$$\frac{v}{x} = \frac{3-t}{1+t}$$

and when $t = 0$, $x = 5$.

(a) Find an expression for x in terms of t. [4]

(b) Find the magnitude of F when $t = 3$. [3]

10. [9231/w24/32/q7]

A particle P of mass m kg is held at rest at a point O and released so that it moves vertically under gravity against a resistive force of magnitude $0.1mv^2$ N, where v m s^{-1} is the velocity of P at time t s.

(a) Find an expression for v in terms of t. [6]

The displacement of P from O at time t s is x m.

(b) Find an expression for v^2 in terms of x. [5]

11. [9231/s23/31/q6]

A particle P moving in a straight line has displacement x m from a fixed point O on the line and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The acceleration of P, in $\mathrm{m\,s^{-2}}$, is given by $6v\sqrt{v+9}$. When $t = 0$, $x = 2$ and $v = 72$.

(a) Find an expression for v in terms of x. [4]

(b) Find an expression for x in terms of t. [5]

12. [9231/s23/33/q6]

A particle of mass m kg falls vertically under gravity, from rest. At time t s, P has fallen x m and has velocity $v\,\mathrm{m\,s^{-1}}$. The only forces acting on P are its weight and a resistance of magnitude $kmgv$ N, where k is a constant.

(a) Find an expression for v in terms of t, g and k. [5]

(b) Given that $k = 0.05$, find, in metres, how far P has fallen when its speed is $12\,\mathrm{m\,s^{-1}}$. [5]

13. [9231/w23/31/q2]

A ball of mass 2 kg is projected vertically downwards with speed $5\,\mathrm{m\,s^{-1}}$ through a liquid. At time t s after projection, the velocity of the ball is $v\,\mathrm{m\,s^{-1}}$ and its displacement from its starting point is x m. The forces acting on the ball are its weight and a resistive force of magnitude $0.2v^2$ N.

(a) Find an expression for v in terms of t. [6]

(b) Deduce what happens to v for large values of t. [1]

14. [9231/w23/32/q2]

A particle P of mass 0.5 kg moves in a straight line. At time t s the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from a fixed point O on the line is x m. The only forces acting on P are a force of magnitude $\frac{150}{(x+1)^2}$ N in the direction of increasing displacement and a resistive force of magnitude $\frac{450}{(x+1)^3}$ N. When $t = 0$, $x = 0$ and $v = 20$.

Find v in terms of x, giving your answer in the form $v = \frac{Ax+B}{(x+1)}$, where A and B are constants to be determined. [6]

15. [9231/s22/31/q3]

A particle P is moving in a horizontal straight line. Initially P is at the point O on the line and is moving with velocity $25\,\mathrm{m\,s^{-1}}$. At time t s after passing through O, the acceleration of P is $\dfrac{4000}{(5t+4)^3}\,\mathrm{m\,s^{-2}}$ in the direction PO. The displacement of P from O at time t is x m.

Find an expression for x in terms of t. [5]

16. [9231/s22/33/q5]

A particle P of mass 4 kg is moving in a horizontal straight line. At time t s the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and the displacement of P from a fixed point O on the line is x m. The only force acting on P is a resistive force of magnitude $(4e^{-x}+12)e^{-x}$ N. When $t=0$, $x=0$ and $v=4$.

(a) Show by integration that $v=\dfrac{1+3e^{x}}{e^{x}}$. [4]

(b) Find an expression for x in terms of t. [4]

17. [9231/w22/31/q4]

A particle of mass 0.5 kg moves along a horizontal straight line. Its velocity is $v\,\text{m}\,\text{s}^{-1}$ at time t s. The forces acting on the particle are a driving force of magnitude 50 N and a resistance of magnitude $2v^2$ N. The initial velocity of the particle is $3\,\text{m}\,\text{s}^{-1}$.

(a) Find an expression for v in terms of t. [7]

(b) Deduce the limiting value of v. [1]

18. [9231/w22/32/q4]

A particle P of mass 5 kg moves along a horizontal straight line. At time t s, the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from a fixed point O on the line is x m. The forces acting on P are a force of magnitude $\frac{500}{v}$ N in the direction OP and a resistive force of magnitude $\frac{1}{2}v^2$ N. When $t = 0$, $x = 0$ and $v = 5$.

(a) Find an expression for v in terms of x. [6]

(b) State the value that the speed approaches for large values of x. [1]

19. [9231/s21/31/q1]

A particle P of mass 1 kg is moving along a straight line against a resistive force of magnitude $\frac{10\sqrt{v}}{(t+1)^2}$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s. When $t = 0$, $v = 25$.

Find an expression for v in terms of t. [5]

20. [9231/s21/33/q5]

A particle P of mass m kg is projected vertically upwards from a point O, with speed $20\,\mathrm{m\,s^{-1}}$, and moves under gravity. There is a resistive force of magnitude $2mv$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s after projection.

(a) Find an expression for v in terms of t, while P is moving upwards. [6]

The displacement of P from O is x m at time t s.

(b) Find an expression for x in terms of t, while P is moving upwards. [2]

(c) Find, correct to 3 significant figures, the greatest height above O reached by P. [2]

21. [9231/w21/31/q2]

A particle P of mass m kg moves along a horizontal straight line with acceleration $a \text{ m s}^{-2}$ given by

$$a = \frac{v(1-2t^2)}{t},$$

where $v \text{ m s}^{-1}$ is the velocity of P at time t s.

(a) Find an expression for v in terms of t and an arbitrary constant. [3]

(b) Given that $a = 5$ when $t = 1$, find an expression, in terms of m and t, for the horizontal force acting on P at time t. [3]

22. [9231/w21/32/q6]

A particle P of mass 2 kg moves along a horizontal straight line. The point O is a fixed point on this line. At time t s the velocity of P is v m s^{-1} and the displacement of P from O is x m.

A force of magnitude $\left(8x - \dfrac{128}{x^3}\right)$ N acts on P in the direction OP. When $t = 0$, $x = 8$ and $v = -15$.

(a) Show that $v = -\dfrac{2}{x}(x^2 - 4)$. [5]

(b) Find an expression for x in terms of t. [4]

23. [9231/s20/31/q5]

A particle P is moving along a straight line with acceleration $3ku - kv$ where v is its velocity at time t, u is its initial velocity and k is a constant. The velocity and acceleration of P are both in the direction of increasing displacement from the initial position.

(a) Find the time taken for P to achieve a velocity of $2u$. [3]

(b) Find an expression for the displacement of P from its initial position when its velocity is $2u$. [5]

24. [9231/s20/33/q2]

A particle Q of mass m kg falls from rest under gravity. The motion of Q is resisted by a force of magnitude mkv N, where v m s^{-1} is the speed of Q at time t s and k is a positive constant.

Find an expression for v in terms of g, k and t. [6]

25. [9231/w20/31/q7]

A particle P moving in a straight line has displacement x m from a fixed point O on the line at time t s. The acceleration of P, in m s^{-2}, is given by $\frac{200}{x^2} - \frac{100}{x^3}$ for $x > 0$. When $t = 0$, $x = 1$ and P has velocity 10 m s^{-1} directed towards O.

(a) Show that the velocity v m s^{-1} of P is given by $v = \frac{10(1-2x)}{x}$. [5]

(b) Show that x and t are related by the equation $e^{-40t} = (2x-1)e^{2x-2}$ and deduce what happens to x as t becomes large. [5]

26. [9231/w20/32/q7]

A particle P of mass m kg moves in a horizontal straight line against a resistive force of magnitude mkv^2 N, where v m s^{-1} is the speed of P after it has moved a distance x m and k is a positive constant. The initial speed of P is u m s^{-1}.

(a) Show that $x = \frac{1}{k}\ln 2$ when $v = \frac{1}{2}u$. [4]

Beginning at the instant when the speed of P is $\frac{1}{2}u$, an additional force acts on P. This force has magnitude $\frac{5m}{v}$ N and acts in the direction of increasing x.

(b) Show that when the speed of P has increased again to u m s^{-1}, the total distance travelled by P is given by an expression of the form

$$\frac{1}{3k}\ln\left(\frac{A - ku^3}{B - ku^3}\right),$$

stating the values of the constants A and B. [7]

27. [9709/m19/52/q7]

A particle P is projected horizontally from a point O on a rough horizontal surface. The coefficient of friction between the particle and the surface is 0.2. A horizontal force of magnitude $0.06t$ N directed away from O acts on P, where t s is the time after projection. P comes to rest when $t = 4$.

(i) The particle begins to move again when $t = 8$. Show that the mass of P is 0.24 kg. [2]

(ii) Show that, for $0 \leqslant t \leqslant 4$, $\frac{dv}{dt} = 0.25t - 2$, and find the speed of projection of P. [5]

(iii) Find the distance from O at which P comes to rest. [4]

28. [9709/s19/51/q7]

A particle P of mass 0.5 kg is attached to a fixed point O by a light elastic string of natural length 1 m and modulus of elasticity 16 N. The particle P is projected vertically upwards from O with speed $6\,\text{m}\,\text{s}^{-1}$. A resisting force of magnitude $0.1x^2$ N acts on P when P has displacement x m above O. After projection the upwards velocity of P is $v\,\text{m}\,\text{s}^{-1}$.

(i) Show that, before the string becomes taut, $v\dfrac{dv}{dx} = -10 - 0.2x^2$. [2]

(ii) Find the velocity of P at the instant the string becomes taut. [4]

(iii) Find an expression for the acceleration of P while it is moving upwards after the string becomes taut. [2]

(iv) Verify that P comes to instantaneous rest before the extension of the string is 0.5 m. [4]

29. [9709/s19/52/q4]

A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The other end of the string is attached to a fixed point O. The particle P is released from rest at the point 0.8 m vertically below O. When the extension of the string is x m, the downwards velocity of P is $v\,\mathrm{m\,s^{-1}}$ and a force of magnitude $25x^2$ N opposes the motion of P.

(i) Show that, when P is moving downwards, $v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 10 - 40x - 50x^2$. [2]

(ii) For the instant when P has its greatest downwards speed, find the kinetic energy of P and the elastic potential energy stored in the string. [6]

30. [9709/w19/51/q3]

A smooth horizontal surface has two fixed points O and A which are 0.8 m apart. A particle P of mass 0.25 kg is projected with velocity $3\,\mathrm{m\,s^{-1}}$ horizontally from A in the direction away from O. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ when the displacement of P from O is x m. A force of magnitude kv^2x^{-2} N opposes the motion of P.

(i) Show that $v\dfrac{\mathrm{d}v}{\mathrm{d}x} = -4kv^2x^{-2}$. [1]

(ii) Express v in terms of k and x. [5]

31. [9709/w19/52/q6]

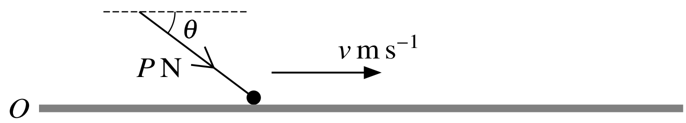
A particle P of mass 0.2 kg is projected horizontally from a fixed point O on a smooth horizontal surface. When the displacement of P from O is x m the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A horizontal force of variable magnitude $0.09\sqrt{x}$ N directed away from O acts on P. An additional force of constant magnitude 0.3 N directed towards O acts on P.

(i) Show that $v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0.45\sqrt{x} - 1.5$. [2]

(ii) Find the value of x for which the acceleration of P is zero. [2]

(iii) Given that the minimum value of v is positive, find the set of possible values for the speed of projection. [5]

32. [9709/m18/52/q6]



A small object of mass 0.2 kg rests at a point O on a rough horizontal surface. The coefficient of friction between the object and the surface is 0.5. A force of magnitude P N acting at an angle θ below the horizontal is applied to the object. The velocity of the object is $v\,\text{m s}^{-1}$ away from O at time t s after the force begins to act (see diagram). It is given that $\tan\theta = \frac{3}{4}$ and that $P = 0.4t$ for $0 \leqslant t \leqslant 8$.

(i) Find the value of t when the object starts to move. [3]

(ii) Show that, when the force is acting and the object is in motion, $\dfrac{dv}{dt} = t - 5$. [2]

When $t = 8$ the force of magnitude P N ceases to act.

(iii) Find the distance travelled by the object after $t = 8$ before it comes to rest. [5]

33. [9709/s18/51/q3]

A particle P of mass 0.4 kg is projected horizontally along a smooth horizontal plane from a point O. At time t s after projection the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A force of magnitude $0.8t$ N directed away from O acts on P and a force of magnitude $2e^{-t}$ N opposes the motion of P.

(i) Show that $\frac{dv}{dt} = 2t - 5e^{-t}$. [2]

(ii) Given that $v = 8$ when $t = 1$, express v in terms of t. [3]

(iii) Find the speed of projection of P. [2]

34. [9709/s18/52/q7]

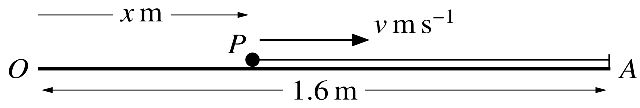
A particle P of mass 0.2 kg is released from rest at a point O above horizontal ground. At time t s after its release the velocity of P is $v\,\mathrm{m\,s^{-1}}$ downwards. A vertically downwards force of magnitude $0.6t$ N acts on P. A vertically upwards force of magnitude ke^{-t} N, where k is a constant, also acts on P.

(i) Show that $\dfrac{dv}{dt} = 10 - 5ke^{-t} + 3t$. [2]

(ii) Find the greatest value of k for which P does not initially move upwards. [3]

(iii) Given that $k = 1$, and that P strikes the ground when $t = 2$, find the height of O above the ground. [5]

35. [9709/w18/51/q4]



A particle P of mass 0.5 kg is projected along a smooth horizontal surface towards a fixed point A. Initially P is at a point O on the surface, and after projection, P has a displacement from O of x m and velocity v m s^{-1}. The particle P is connected to A by a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The distance OA is 1.6 m (see diagram). The motion of P is resisted by a force of magnitude $24x^2$ N.

(i) Show that $v\dfrac{dv}{dx} = 32 - 40x - 48x^2$ while P is in motion and the string is stretched. [3]

The maximum value of v is 4.5.

(ii) Find the initial value of v. [5]

36. [9709/w18/52/q3]

A particle P of mass 0.4 kg is projected horizontally along a smooth horizontal plane from a point O. After projection the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from O is x m. A force of magnitude $8x$ N directed away from O acts on P and a force of magnitude $(2e^{-x} + 4)$ N opposes the motion of P. One end of a light elastic string of natural length 0.5 m is attached to O and the other end of the string is attached to P.

(i) Show that $v\dfrac{dv}{dx} = 20x - 10 - 5e^{-x}$ before the elastic string becomes taut. [2]

(ii) Given that the initial velocity of P is $6\,\mathrm{m\,s^{-1}}$, find v when the string first becomes taut. [3]

When the string is taut, the acceleration of P is proportional to e^{-x}.

(iii) Find the modulus of elasticity of the string. [2]

37. [9709/m17/52/q6]

O and A are fixed points on a rough horizontal surface, with $OA = 1$ m. A particle P of mass 0.4 kg is projected horizontally with speed $U\,\mathrm{m\,s^{-1}}$ from A in the direction OA and moves in a straight line. After projection, when the displacement of P from O is x m, the velocity of P is $v\,\mathrm{m\,s^{-1}}$. The coefficient of friction between the surface and P is 0.4. A force of magnitude $\frac{0.8}{x}$ N acts on P in the direction PO.

(i) Show that, while the particle is in motion, $v\frac{\mathrm{d}v}{\mathrm{d}x} = -4 - \frac{2}{x}$. [3]

It is given that P comes to instantaneous rest between $x = 2.0$ and $x = 2.1$.

(ii) Find the set of possible values of U. [5]

38. [9709/s17/51/q7]

A particle P of mass 0.5 kg is at rest at a point O on a rough horizontal surface. At time $t = 0$, where t is in seconds, a horizontal force acting in a fixed direction is applied to P. At time t s the magnitude of the force is $0.6t^2$ N and the velocity of P away from O is $v \text{ m s}^{-1}$. It is given that P remains at rest at O until $t = 0.5$.

(i) Calculate the coefficient of friction between P and the surface, and show that

$$\frac{dv}{dt} = 1.2t^2 - 0.3 \quad \text{for } t > 0.5.$$ [3]

(ii) Express v in terms of t for $t > 0.5$. [3]

(iii) Find the displacement of P from O when $t = 1.2$. [3]

39. [9709/s17/52/q4]

A small object of mass 0.4 kg is released from rest at a point 8 m above the ground. The object descends vertically and when its downwards displacement from its initial position is x m the object has velocity $v\,\text{m s}^{-1}$. While the object is moving, a force of magnitude $0.2v^2$ N opposes the motion.

(i) Show that $v\dfrac{dv}{dx} = 10 - 0.5v^2$. [2]

(ii) Express v in terms of x. [4]

(iii) Find the increase in the value of v during the final 4 m of the descent of the object. [2]

40. [9709/w17/51/q7]

A particle P of mass 0.2 kg is released from rest at a point O on a rough plane inclined at 60° to the horizontal, and travels down a line of greatest slope. The coefficient of friction between P and the plane is 0.3. A force of magnitude $0.6x$ N acts on P in the direction PO, where x m is the displacement of P from O.

(i) Show that $v\frac{\mathrm{d}v}{\mathrm{d}x} = 5\sqrt{3} - 1.5 - 3x$, where $v\,\mathrm{m\,s^{-1}}$ is the velocity of P at a displacement x m from O. [3]

(ii) Find the value of x for which P reaches its maximum velocity, and calculate this maximum velocity. [4]

(iii) Calculate the magnitude of the acceleration of P immediately after it has first come to instantaneous rest. [4]

41. [9709/w17/52/q1]

A particle P of mass 0.2 kg is released from rest at a point O on a smooth horizontal surface. A horizontal force of magnitude $t\mathrm{e}^{-v}$ N directed away from O acts on P, where $v\ \mathrm{m\,s^{-1}}$ is the velocity of P at time t s after release. Find the velocity of P when $t = 2$. [4]

42. [9709/w17/52/q3]

A particle P of mass 0.4 kg is released from rest at a point O on a smooth plane inclined at 30° to the horizontal. P moves down the line of greatest slope through O. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ when its displacement from O is x m. A retarding force of magnitude $0.2v^2$ N acts on P in the direction PO.

(i) Show that $v\dfrac{dv}{dx} = 5 - 0.5v^2$. [2]

(ii) Express v in terms of x. [4]

43. [9709/m16/52/q6]

A particle P of mass 0.2 kg is released from rest at a point O on a plane inclined at 30° to the horizontal. At time t s after its release, P has velocity $v\,\mathrm{m\,s^{-1}}$ and displacement x m down the plane from O. The coefficient of friction between P and the plane increases as P moves down the plane, and equals $0.1x^2$.

(i) Show that $2v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 10 - (\sqrt{3})x^2$. [2]

(ii) Calculate the maximum speed of P. [5]

(iii) Find the value of x at the point where P comes to rest. [2]

44. [9709/s16/51/q3]

A particle P of mass 0.4 kg is released from rest at a point O on a smooth plane inclined at 30° to the horizontal. When the displacement of P from O is x m down the plane, the velocity of P is v m s^{-1}. A force of magnitude $0.8e^{-x}$ N acts on P up the plane along the line of greatest slope through O.

(i) Show that $v\dfrac{dv}{dx} = 5 - 2e^{-x}$. [2]

(ii) Find v when $x = 0.6$. [4]

45. [9709/s16/52/q5]

A particle P of mass 0.4 kg is placed at rest at a point A on a rough horizontal surface. A horizontal force, directed away from A and with magnitude $0.6t$ N, acts on P, where t s is the time after P is placed at A. The coefficient of friction between P and the surface is 0.3, and P has displacement from A of x m at time t s.

(i) Show that P starts to move when $t = 2$. Show also that when P is in motion it has acceleration $(1.5t - 3)\,\text{m s}^{-2}$. [3]

(ii) Express the velocity of P in terms of t, for $t \geqslant 2$. [4]

(iii) Express x in terms of t, for $t \geqslant 2$. [3]

46. [9709/w16/51/q3]

A small block B of mass 0.25 kg is released from rest at a point O on a smooth horizontal surface. After its release the velocity of B is $v\,\mathrm{m\,s^{-1}}$ when its displacement is x m from O. The force acting on B has magnitude $(2 + 0.3x^2)$ N and is directed horizontally away from O.

(i) Show that $v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1.2x^2 + 8$. [2]

(ii) Find the velocity of B when $x = 1.5$. [3]

An extra force acts on B after $x = 1.5$. It is given that, when $x > 1.5$,

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = 1.2x^2 + 6 - 3x.$$

(iii) Find the magnitude of this extra force and state the direction in which it acts. [2]

47. [9709/w16/52/q5]

A particle P of mass 0.4 kg is released from rest at a point O on a smooth plane inclined at 30° to the horizontal. A force of magnitude $3e^{-t}$ N directed up a line of greatest slope acts on P, where t s is the time after release.

(i) Show that $\frac{dv}{dt} = 7.5e^{-t} - 5$, where v m s^{-1} is the velocity of P up the plane at time t s. [2]

(ii) Express v in terms of t. [3]

(iii) Find the distance of P from O when v has its maximum value. [3]

48. [9709/s15/51/q6]

A particle P of mass 0.1 kg moves with decreasing speed in a straight line on a smooth horizontal surface. A horizontal resisting force of magnitude $0.2e^{-x}$ N acts on P, where x m is the displacement of P from a fixed point O on the line. The velocity of P is $v\,\text{m s}^{-1}$ when its displacement from O is x m.

(i) Show that

$$v\frac{dv}{dx} = ke^{-x},$$

where k is a constant to be found. [2]

P passes through O with velocity $2.2\,\text{m s}^{-1}$.

(ii) Calculate the value of x at the instant when the velocity of P is $2\,\text{m s}^{-1}$. [4]

(iii) Show that the speed of P does not fall below $0.917\,\text{m s}^{-1}$, correct to 3 significant figures. [2]

49. [9709/s15/52/q7]

A force of magnitude $0.4t$ N, applied at an angle of 30° above the horizontal, acts on a particle P, where t s is the time since the force starts to act. P is at rest on rough horizontal ground when $t = 0$. The mass of P is 0.2 kg and the coefficient of friction between P and the ground is μ.

(i) Given that P is about to slip when $t = 2$, find μ and the value of t for the instant when P loses contact with the ground. [5]

(ii) While P is moving on the ground, it has velocity v m s^{-1} at time t s. Show that

$$\frac{dv}{dt} = 2.165t - 4.330,$$

where the coefficients are correct to 4 significant figures. [3]

(iii) Calculate the speed of P when it loses contact with the ground. [4]

50. [9709/s15/53/q6]

A cyclist and her bicycle have a total mass of 60 kg. The cyclist rides in a horizontal straight line, and exerts a constant force in the direction of motion of 150 N. The motion is opposed by a resistance of magnitude $12v$ N, where v m s^{-1} is the cyclist's speed at time t s after passing through a fixed point A.

(i) Show that $5\frac{\mathrm{d}v}{\mathrm{d}t} = 12.5 - v$. [2]

(ii) Given that the cyclist passes through A with speed 11.5 m s^{-1}, solve this differential equation to show that $v = 12.5 - \mathrm{e}^{-0.2t}$. [4]

(iii) Express the displacement of the cyclist from A in terms of t. [3]

51. [9709/w15/51/q1]

A particle P moves in a straight line and passes through a point O of the line with velocity $2\,\mathrm{m\,s^{-1}}$. At time t s after passing through O, the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and the acceleration of P is given by $e^{-0.5v}\,\mathrm{m\,s^{-2}}$. Calculate the velocity of P when $t = 1.2$. [4]

52. [9709/w15/51/q3]

A particle P of mass 0.3 kg moves in a straight line on a smooth horizontal surface. P passes through a fixed point O of the line with velocity $8\,\mathrm{m\,s^{-1}}$. A force of magnitude $2x$ N acts on P in the direction PO, where x m is the displacement of P from O.

(i) Show that $v\dfrac{\mathrm{d}v}{\mathrm{d}x} = kx$ and state the value of the constant k. [2]

(ii) Find the value of x at the instant when P comes to instantaneous rest. [3]

53. [9709/w15/53/q5]

A particle P of mass 0.5 kg is projected vertically upwards from a point on a horizontal surface. A resisting force of magnitude $0.02v^2$ N acts on P, where $v\,\text{m}\,\text{s}^{-1}$ is the upward velocity of P when it is a height of x m above the surface. The initial speed of P is $8\,\text{m}\,\text{s}^{-1}$.

(i) Show that, while P is moving upwards, $v\dfrac{dv}{dx} = -10 - 0.04v^2$. [2]

(ii) Find the greatest height of P above the surface. [3]

(iii) Find the speed of P immediately before it strikes the surface after descending. [4]

Chapter 7

Momentum

1. [9231/s25/31/q6]

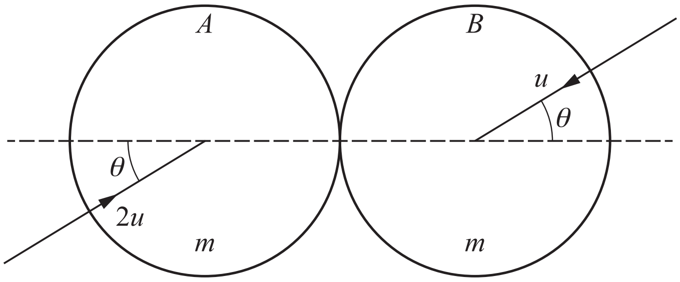
Two uniform smooth spheres A and B of equal radii have masses $2m$ and m respectively. Sphere A is moving in a straight horizontal line with speed u, and sphere B is stationary. Sphere A collides directly with B, and they both then move in the same direction with speeds v_A and v_B respectively. After the collision, the kinetic energy of B is $\frac{9}{2}$ times the kinetic energy of A.

(a) Show that $v_B = \frac{6}{5}u$. [3]

Sphere B then collides with a fixed vertical barrier. Immediately before the collision, the direction of motion of B makes an angle α with the barrier. Immediately after the collision, the direction of motion of B makes an angle β with the barrier. The coefficient of restitution between B and the barrier is $\frac{4}{5}$. As a result of the collision, the velocity of B is reduced to $\frac{12}{25}\sqrt{5}u$.

(b) Find the value of $\sin(\alpha + \beta)$. [6]

2. [9231/s25/33/q6]



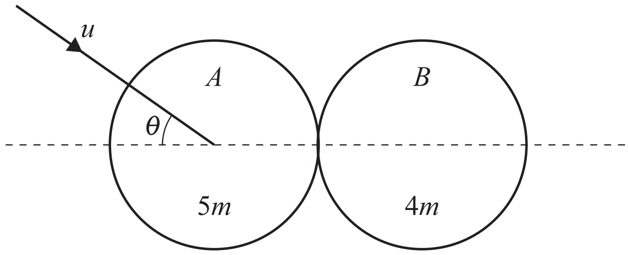
Two identical uniform smooth spheres A and B, each with mass m, are moving on a horizontal surface with speeds $2u$ and u respectively when they collide. Immediately before the collision, the spheres are moving parallel to each other in opposite directions such that their directions of motion each make an angle θ with the line of centres (see diagram). As a result of the collision, B moves in a direction which is perpendicular to its initial direction of motion. The coefficient of restitution between the spheres is e.

(a) Find an expression for $\tan\theta$ in terms of e. [6]

As a result of the collision, A moves in a direction which is perpendicular to the line of centres.

(b) Find the value of θ. [2]

3. [9231/s25/34/q6]



Two uniform smooth spheres A and B of equal radii have masses $5m$ and $4m$ respectively. Sphere A is moving with speed u on a horizontal surface when it collides with sphere B which is at rest. Immediately before the collision, A's direction of motion makes an angle θ with the line of centres (see diagram). The coefficient of restitution between the spheres is e.

(a) Show that the speed of B after the collision is $\frac{5}{9}u(1+e)\cos\theta$. [3]

After the collision the kinetic energy of A is equal to the kinetic energy of B.

(b) Given that $\tan\theta = \frac{2}{3}$, find the value of e. [6]

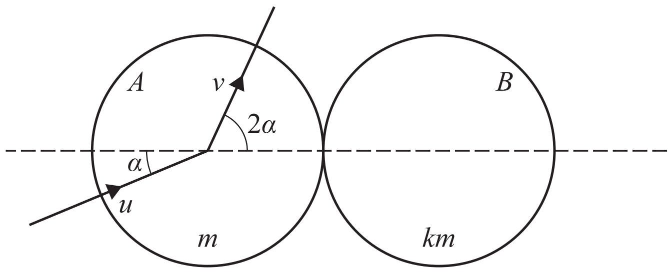
4. [9231/w25/31/q1]

Two uniform smooth spheres A and B of equal radii have masses $4m$ and m respectively. Sphere B is at rest on a smooth horizontal surface. Sphere A is moving on the surface with speed u and collides directly with sphere B. After the collision, the momentum of A is three times the momentum of B.

Find the value of the coefficient of restitution e. [4]

5. [9231/w25/32/q5]

Two uniform smooth spheres, A and B, of equal radii are on a horizontal surface. They have masses of m and km respectively. Sphere A is moving with speed u at an angle α with the line of centres when it collides with sphere B which is stationary. Immediately after the collision, sphere A moves with speed v at an angle 2α with the line of centres (see diagram).



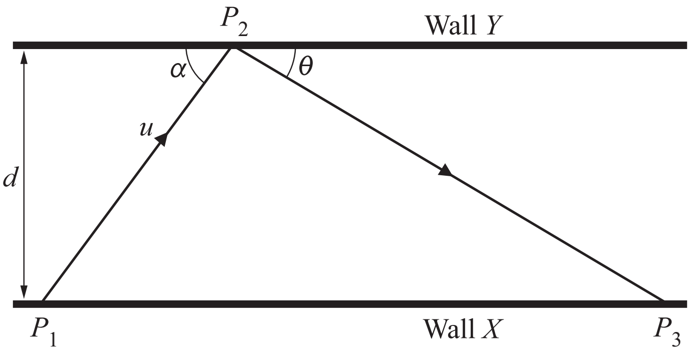
It is given that $\tan\alpha = \frac{3}{4}$.

(a) Find v in terms of u. [2]

(b) Find the coefficient of restitution between the spheres in terms of k. [4]

(c) Find the range of possible values of k. [3]

6. [9231/w25/34/q7]



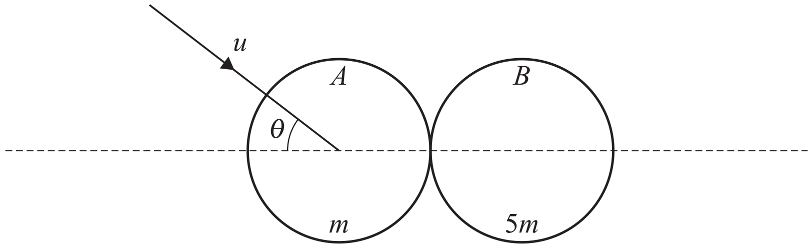
X and Y are two fixed smooth vertical walls on a smooth horizontal surface. The walls are parallel and at a distance d apart. The points P_1, P_2 and P_3 all lie on the surface.

A particle Q is projected horizontally from the point P_1 on Wall X with speed u, and moves along the surface. The particle Q strikes Wall Y at the point P_2. Immediately before the collision, the direction of motion of Q makes an angle α with Wall Y, where $\sin\alpha = \frac{4}{5}$. Immediately after the collision, the direction of motion of Q makes an angle θ with Wall Y. The particle Q then strikes Wall X at the point P_3 (see diagram).

The time that it takes Q to travel the distance P_1P_2 is T. The time that it takes Q to travel the distance P_2P_3 is kT.

Find, in terms of k, the coefficient of restitution between Q and Wall Y. [6]

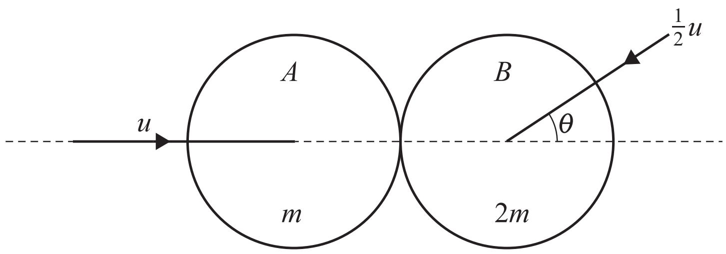
7. [9231/s24/31/q1]



Two smooth uniform spheres A and B of equal radii have masses m and $5m$ respectively. Sphere A is moving on a smooth horizontal surface with speed u when it collides with sphere B which is at rest on the surface. Immediately before the collision, A's direction of motion makes an angle of θ with the line of centres. After the collision, the kinetic energies of A and B are equal. The coefficient of restitution between the spheres is $\frac{1}{2}$.

Find the value of $\tan\theta$. [6]

8. [9231/s24/33/q1]



Two smooth uniform spheres A and B of equal radii have masses m and $2m$ respectively. The two spheres are moving on a smooth horizontal surface when they collide with speeds u and $\frac{1}{2}u$ respectively. Immediately before the collision, A's direction of motion is along the line of centres, and B's direction of motion makes an angle θ with the line of centres (see diagram).

As a result of the collision, the direction of motion of A is reversed and its speed is reduced to $\frac{1}{4}u$. The direction of motion of B again makes an angle θ with the line of centres, but on the opposite side of the line of centres. The speed of B is unchanged.

Find the value of the coefficient of restitution between the spheres. [4]

9. [9231/w24/31/q7]

A particle P is projected with speed u at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ above the horizontal from a point O on a horizontal plane and moves freely under gravity. When P is moving horizontally, it strikes a smooth inclined plane at the point A. This plane is inclined to the horizontal at an angle α, and the line of greatest slope through A lies in the vertical plane through O and A.

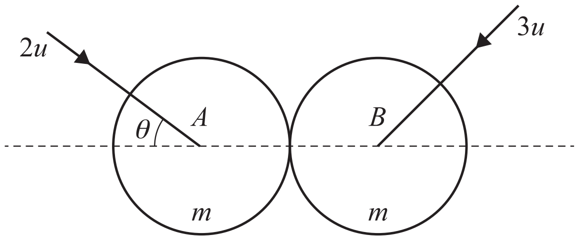
As a result of the impact, P moves vertically upwards. The coefficient of restitution between P and the inclined plane is e.

(a) Show that $e\tan^2\alpha = 1$. [4]

In its subsequent motion, the greatest height reached by P above A is $\frac{3}{16}$ of the vertical height of A above the horizontal plane.

(b) Find the value of e. [6]

10. [9231/w24/32/q3]



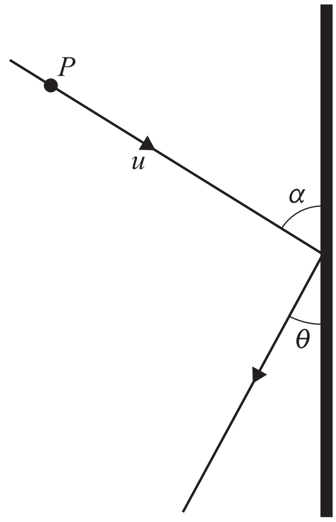
The diagram shows two identical smooth uniform spheres A and B of equal radii and each of mass m. The two spheres are moving on a smooth horizontal surface when they collide with speeds $2u$ and $3u$ respectively. Immediately before the collision, A's direction of motion makes an angle θ with the line of centres and B's direction of motion is perpendicular to that of A. After the collision, B moves perpendicular to the line of centres. The coefficient of restitution between the spheres is $\frac{1}{3}$.

(a) Find the value of $\tan\theta$. [3]

(b) Find the total loss of kinetic energy as a result of the collision. [2]

(c) Find, in degrees, the angle through which the direction of motion of A is deflected as a result of the collision. [2]

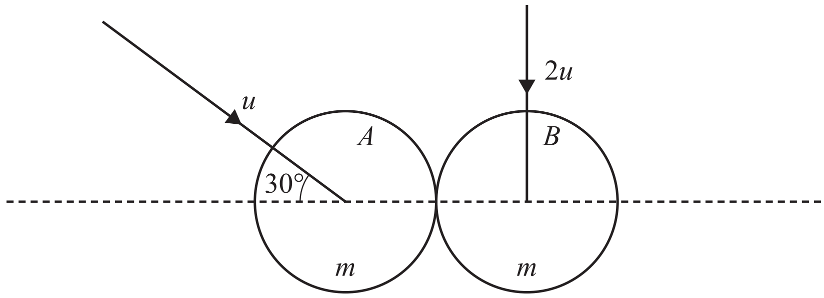
11. [9231/s23/31/q2]



A particle P of mass m is moving with speed u on a fixed smooth horizontal surface. It collides at an angle α with a fixed smooth vertical barrier. After the collision, P moves at an angle θ with the barrier, where $\tan\theta = \frac{1}{2}$ (see diagram). The coefficient of restitution between P and the barrier is e. The particle P loses 20% of its kinetic energy as a result of the collision.

Find the value of e. [5]

12. [9231/s23/33/q4]

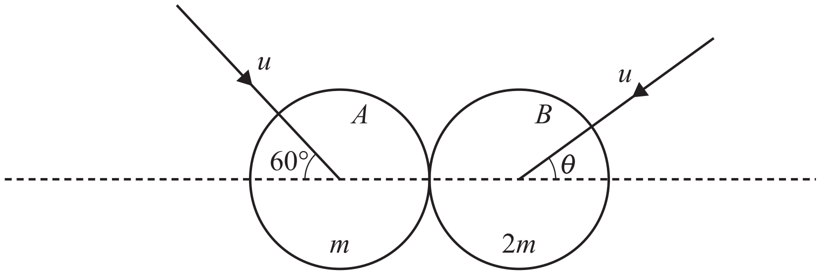


Two identical smooth uniform spheres A and B each have mass m. The two spheres are moving on a smooth horizontal surface when they collide with speeds u and $2u$ respectively. Immediately before the collision, A's direction of motion makes an angle of 30° with the line of centres, and B's direction of motion is perpendicular to the line of centres (see diagram). After the collision, A and B are moving in the same direction. The coefficient of restitution between the spheres is e.

(a) Find the value of e. [5]

(b) Find the loss in the total kinetic energy of the spheres as a result of the collision. [3]

13. [9231/w23/31/q1]



Two uniform smooth spheres A and B of equal radii have masses m and $2m$ respectively. The two spheres are moving with equal speeds u on a smooth horizontal surface when they collide. Immediately before the collision, A's direction of motion makes an angle of 60° with the line of centres, and B's direction of motion makes an angle θ with the line of centres (see diagram). The coefficient of restitution between the spheres is e.

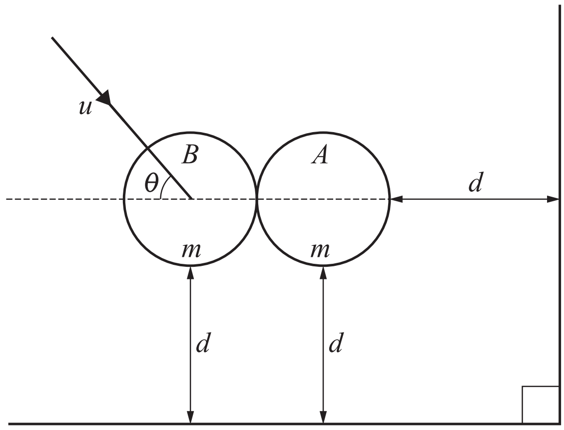
After the collision, the component of the velocity of A along the line of centres is v and B moves perpendicular to the line of centres. Sphere A now has twice as much kinetic energy as sphere B.

(a) Show that $v = \frac{1}{2}u(4\cos\theta - 1)$. [1]

(b) Find the value of $\cos\theta$. [4]

(c) Find the value of e. [2]

14. [9231/w23/32/q4]



Two smooth vertical walls meet at right angles. The smooth sphere *A*, with mass *m*, is at rest on a smooth horizontal surface and is at a distance *d* from each wall. An identical smooth sphere *B* is moving on the horizontal surface with speed *u* at an angle θ with the line of centres when the spheres collide (see diagram). After the collision, the spheres take the same time to reach a wall. The coefficient of restitution between the spheres is $\frac{1}{2}$.

(a) Find the value of $\tan\theta$. [4]

(b) Find the percentage loss in the total kinetic energy of the spheres as a result of this collision. [3]

15. [9231/s22/31/q6]

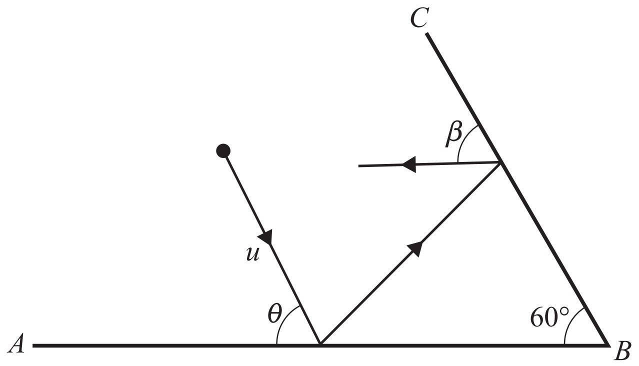
Two uniform smooth spheres A and B of equal radii have masses m and km respectively. The two spheres are on a horizontal surface. Sphere A is travelling with speed u towards sphere B which is at rest. The spheres collide. Immediately before the collision, the direction of motion of A makes an angle α with the line of centres. The coefficient of restitution between the spheres is $\frac{1}{2}$.

(a) Show that the speed of B after the collision is $\dfrac{3u\cos\alpha}{2(1+k)}$ and find also an expression for the speed of A along the line of centres after the collision, in terms of k, u and α. [4]

After the collision, the kinetic energy of A is equal to the kinetic energy of B.

(b) Given that $\tan\alpha = \frac{2}{3}$, find the possible values of k. [5]

16. [9231/s22/33/q6]



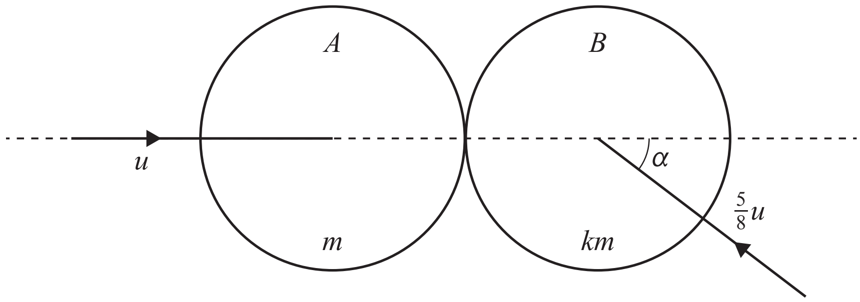
AB and BC are two fixed smooth vertical barriers on a smooth horizontal surface, with angle $ABC = 60°$. A particle of mass m is moving with speed u on the surface. The particle strikes AB at an angle θ with AB. It then strikes BC and rebounds at an angle β with BC (see diagram). The coefficient of restitution between the particle and each barrier is e and $\tan\theta = 2$.

The kinetic energy of the particle after the first collision is 40% of its kinetic energy before the first collision.

(a) Find the value of e. [4]

(b) Find the size of angle β. [4]

17. [9231/w22/31/q6]



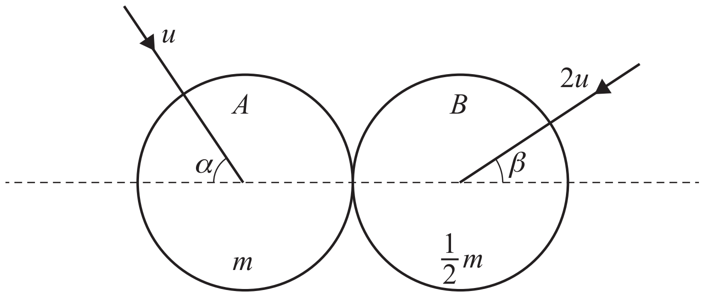
Two uniform smooth spheres A and B of equal radii have masses m and km respectively. The two spheres are moving on a horizontal surface with speeds u and $\frac{5}{8}u$ respectively. Immediately before the spheres collide, A is travelling along the line of centres, and B's direction of motion makes an angle α with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$ and $\tan\alpha = \frac{3}{4}$.

After the collision, the direction of motion of B is perpendicular to the line of centres.

(a) Find the value of k. [4]

(b) Find the loss in the total kinetic energy as a result of the collision. [4]

18. [9231/w22/32/q7]



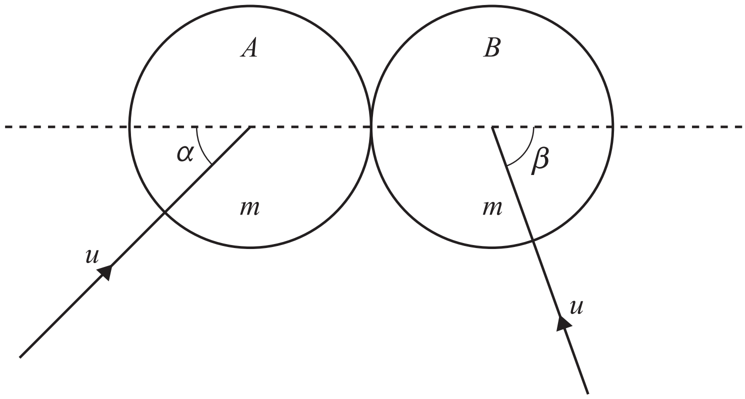
Two uniform smooth spheres A and B of equal radii have masses m and $\frac{1}{2}m$ respectively. The two spheres are moving on a horizontal surface when they collide. Immediately before the collision, sphere A is travelling with speed u and its direction of motion makes an angle α with the line of centres. Sphere B is travelling with speed $2u$ and its direction of motion makes an angle β with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{5}{8}$ and $\alpha + \beta = 90°$.

(a) Find the component of the velocity of B parallel to the line of centres after the collision, giving your answer in terms of u and α. [4]

The direction of motion of B after the collision is parallel to the direction of motion of A before the collision.

(b) Find the value of $\tan\alpha$. [5]

19. [9231/s21/31/q6]



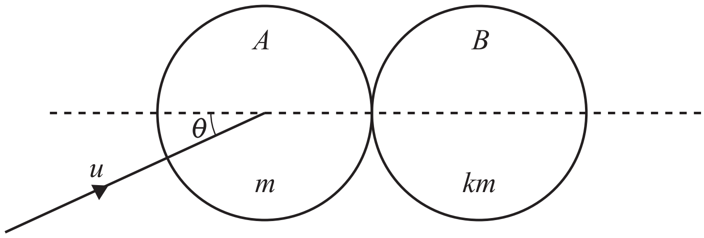
Two uniform smooth spheres A and B of equal radii each have mass m. The two spheres are each moving with speed u on a horizontal surface when they collide. Immediately before the collision, A's direction of motion makes an angle α with the line of centres, and B's direction of motion makes an angle β with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$ and $2\cos\beta = \cos\alpha$.

(a) Show that the direction of motion of A after the collision is perpendicular to the line of centres. [4]

The total kinetic energy of the spheres after the collision is $\frac{3}{4}mu^2$.

(b) Find the value of α. [4]

20. [9231/s21/33/q6]



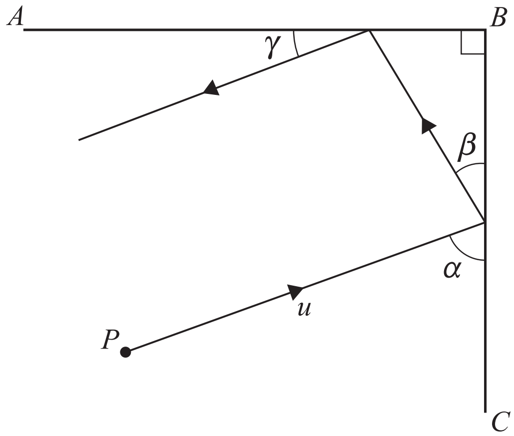
Two uniform smooth spheres A and B of equal radii have masses m and km respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides with sphere B which is at rest. Immediately before the collision, A's direction of motion makes an angle θ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$.

(a) Show that the speed of B after the collision is $\dfrac{4u\cos\theta}{3(1+k)}$. [3]

70% of the total kinetic energy of the spheres is lost as a result of the collision.

(b) Given that $\tan\theta = \frac{1}{3}$, find the value of k. [6]

21. [9231/w21/31/q7]



The smooth vertical walls AB and CB are at right angles to each other. A particle P is moving with speed u on a smooth horizontal floor and strikes the wall CB at an angle α. It rebounds at an angle β to the wall CB. The particle then strikes the wall AB and rebounds at an angle γ to that wall (see diagram). The coefficient of restitution between each wall and P is e.

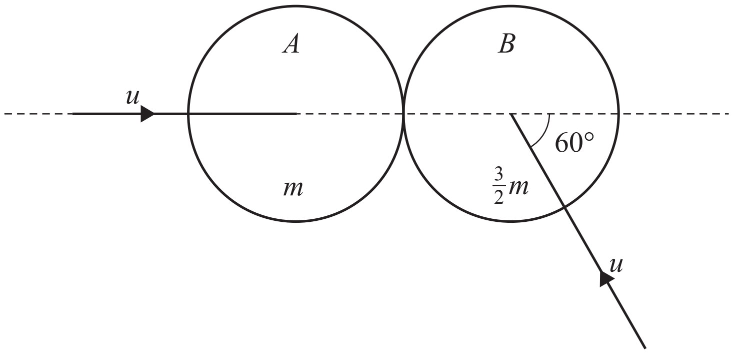
(a) Show that $\tan\beta = e\tan\alpha$. [3]

(b) Express γ in terms of α and explain what this result means about the final direction of motion of P. [4]

As a result of the two impacts the particle loses $\frac{8}{9}$ of its initial kinetic energy.

(c) Given that $\alpha + \beta = 90°$, find the value of e and the value of $\tan\alpha$. [4]

22. [9231/w21/32/q5]



Two uniform smooth spheres A and B of equal radii have masses m and $\frac{3}{2}m$ respectively. The two spheres are each moving with speed u on a horizontal surface when they collide. Immediately before the collision A's direction of motion is along the line of centres, and B's direction of motion makes an angle of 60° with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$.

(a) Find the angle through which the direction of motion of B is deflected by the collision. [6]

(b) Find the loss in the total kinetic energy of the system as a result of the collision. [3]

23. [9231/s20/31/q6]

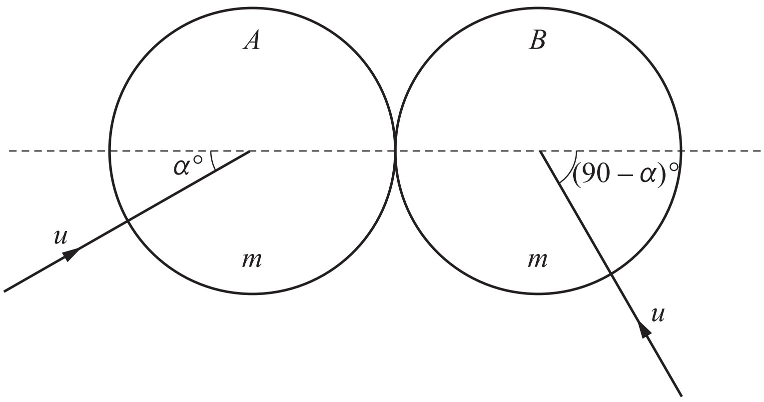
A particle P of mass m is moving with speed u on a fixed smooth horizontal surface. The particle strikes a fixed vertical barrier. At the instant of impact the direction of motion of P makes an angle α with the barrier. The coefficient of restitution between P and the barrier is e. As a result of the impact, the direction of motion of P is turned through 90°.

(a) Show that $\tan^2\alpha = \frac{1}{e}$. [3]

The particle P loses two-thirds of its kinetic energy in the impact.

(b) Find the value of α and the value of e. [5]

24. [9231/s20/33/q5]



Two uniform smooth spheres A and B of equal radii each have mass m. The two spheres are each moving with speed u on a horizontal surface when they collide. Immediately before the collision A's direction of motion makes an angle of $\alpha°$ with the line of centres, and B's direction of motion is perpendicular to that of A (see diagram). The coefficient of restitution between the spheres is e.

Immediately after the collision, B moves in a direction at right angles to the line of centres.

(a) Show that $\tan\alpha = \dfrac{1+e}{1-e}$. [4]

(b) Given that $\tan\alpha = 2$, find the speed of A after the collision. [4]

25. [9231/w20/31/q6]

Two smooth spheres A and B have equal radii and masses m and $2m$ respectively. Sphere B is at rest on a smooth horizontal floor. Sphere A is moving on the floor with velocity u and collides directly with B. The coefficient of restitution between the spheres is e.

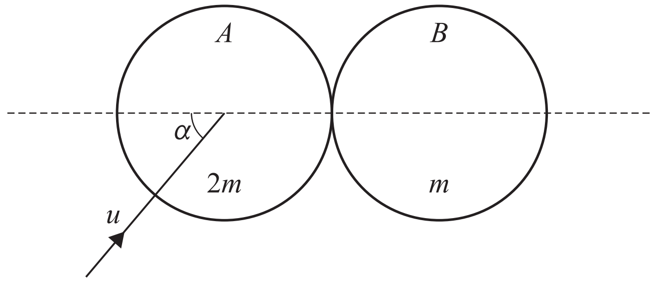
(a) Find, in terms of u and e, the velocities of A and B after the collision. [3]

Subsequently, B collides with a fixed vertical wall which makes an angle θ with the direction of motion of B, where $\tan\theta = \frac{3}{4}$.

The coefficient of restitution between B and the wall is $\frac{2}{3}$. Immediately after B collides with the wall, the kinetic energy of A is $\frac{5}{32}$ of the kinetic energy of B.

(b) Find the possible values of e. [7]

26. [9231/w20/32/q2]



Two uniform smooth spheres A and B of equal radii have masses $2m$ and m respectively. Sphere B is at rest on a smooth horizontal surface. Sphere A is moving on the surface with speed u and collides with B. Immediately before the collision, the direction of motion of A makes an angle α with the line of centres of the spheres, where $\tan\alpha = \frac{4}{3}$ (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$.

Find the speed of A after the collision. [5]

27. [9231/s19/21/q3]

Three uniform small spheres A, B and C have equal radii and masses $2m$, $4m$ and m respectively. The spheres are moving in a straight line on a smooth horizontal surface, with B between A and C. The coefficient of restitution between each pair of spheres is e. Spheres A and B are moving towards each other with speeds $2u$ and u respectively. The first collision is between A and B.

(i) Find the velocities of A and B after this collision. [3]

Sphere C is moving towards B with speed $\frac{4}{3}u$ and now collides with it. As a result of this collision, B is brought to rest.

(ii) Find the value of e. [4]

(iii) Find the total kinetic energy lost by the three spheres as a result of the two collisions. [3]

28. [9231/s19/23/q3]

Three uniform small spheres A, B and C have equal radii and masses $3m$, m and m respectively. The spheres are at rest in a straight line on a smooth horizontal surface, with B between A and C. The coefficient of restitution between each pair of spheres is e. Sphere A is projected directly towards B with speed u.

(i) Find, in terms of u and e, expressions for the speeds of A, B and C after the first two collisions. [6]

(ii) Given that A and C are moving with equal speeds after these two collisions, find the value of e. [3]

29. [9231/w19/21/q3]

Three uniform small spheres A, B and C have equal radii and masses $5m$, $5m$ and $3m$ respectively. The spheres are at rest on a smooth horizontal surface, in a straight line, with B between A and C. The coefficient of restitution between each pair of spheres is e. Sphere A is projected directly towards B with speed u.

(i) Show that the speed of A after its collision with B is $\frac{1}{2}u(1-e)$ and find the speed of B. [3]

Sphere B now collides with sphere C. Subsequently there are no further collisions between any of the spheres.

(ii) Find the set of possible values of e. [6]

30. [9231/s18/21/q1]

A bullet of mass m kg is fired horizontally into a fixed vertical block of material. It enters the block horizontally with speed $250\,\mathrm{m\,s^{-1}}$ and emerges horizontally with speed $70\,\mathrm{m\,s^{-1}}$ after 0.04 s. The block offers a constant horizontal resisting force of magnitude 450 N. Find the value of m. [3]

31. [9231/s18/21/q3]

Two identical uniform small spheres A and B, each of mass m, are moving towards each other in a straight line on a smooth horizontal surface. Their speeds are u and ku respectively, and they collide directly. The coefficient of restitution between the spheres is e. Sphere B is brought to rest by the collision.

(i) Show that $e = \dfrac{k-1}{k+1}$. [3]

(ii) Given that 60% of the total initial kinetic energy is lost in the collision, find the values of k and e. [6]

32. [9231/s18/23/q2]

Two uniform small spheres A and B have equal radii and masses $4m$ and m respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is at rest. The coefficient of restitution between the spheres is e.

(i) Show that after the collision A moves with speed $\frac{1}{5}u(4 - e)$ and find the speed of B. [4]

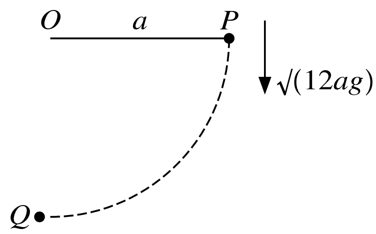
Sphere B continues to move until it collides with a fixed smooth vertical barrier which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the barrier is $\frac{3}{4}e$. After this collision, the speeds of A and B are equal.

(ii) Find the value of e. [3]

The spheres A and B now collide directly again.

(iii) Determine whether sphere B collides with the barrier for a second time. [2]

33. [9231/s18/23/q5]



A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held with the string taut and horizontal. It is projected downwards with speed $\sqrt{(12ag)}$. At the lowest point of its motion, P collides directly with a particle Q of mass km which is at rest (see diagram). In the collision, P and Q coalesce. The tension in the string immediately after the collision is half of its value immediately before the collision. Find the possible values of k. [11]

34. [9231/w18/21/q2]

Two uniform small smooth spheres A and B have equal radii and masses $5m$ and $2m$ respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is moving towards it with speed $2u$. The coefficient of restitution between the spheres is e.

(i) Show that the speed of B after the collision is $\frac{1}{7}u(1 + 15e)$ and find an expression for the speed of A. [4]

In the collision, the speed of A is halved and its direction of motion is reversed.

(ii) Find the value of e. [2]

(iii) For this collision, find the ratio of the loss of kinetic energy of A to the loss of kinetic energy of B. [3]

35. [9231/w18/22/q2]

Two uniform small smooth spheres A and B have equal radii and masses $2m$ and m respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is at rest. The coefficient of restitution between the spheres is $\frac{2}{3}$.

(i) Find, in terms of u, the speeds of A and B after this collision. [4]

Sphere B is initially at a distance d from a fixed smooth vertical wall which is perpendicular to the direction of motion of A. The coefficient of restitution between B and the wall is $\frac{1}{2}$.

(ii) Find, in terms of d and u, the time that elapses between the first and second collisions between A and B. [5]

36. [9231/s17/21/q1]

A bullet of mass 0.08 kg is fired horizontally into a fixed vertical barrier. It enters the barrier horizontally with speed 300 m s^{-1} and emerges horizontally after 0.02 s. There is a constant horizontal resisting force of magnitude 1000 N. Find the speed with which the bullet emerges from the barrier.

[3]

37. [9231/s17/21/q3]

Two uniform small smooth spheres A and B have equal radii and masses $3m$ and m respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is at rest. The coefficient of restitution between the spheres is e.

(i) Find, in terms of u and e, expressions for the velocities of A and B after the collision. [3]

Sphere B continues to move until it strikes a fixed smooth vertical barrier which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the barrier is $\frac{3}{4}$. When the spheres subsequently collide, A is brought to rest.

(ii) Find the value of e. [7]

38. [9231/s17/23/q3]

Two uniform small smooth spheres A and B have equal radii and each has mass m. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is at rest. The coefficient of restitution between the spheres is $\frac{2}{3}$. Sphere B is initially at a distance d from a fixed smooth vertical wall which is perpendicular to the direction of motion of A. The coefficient of restitution between B and the wall is $\frac{1}{3}$.

(i) Show that the speed of B after its collision with the wall is $\frac{5}{18}u$. [4]

(ii) Find the distance of B from the wall when it collides with A for the second time. [6]

39. [9231/w17/21/q3]

Three uniform small smooth spheres A, B and C have equal radii and masses m, km and m respectively, where k is a constant. The spheres are moving in the same direction along a straight line on a smooth horizontal surface, with B between A and C. The speeds of A, B and C are $2u$, u and $\frac{4}{3}u$ respectively. The coefficient of restitution between any pair of the spheres is $\frac{1}{2}$. After sphere A has collided with sphere B, sphere B collides with sphere C.

(i) Find an inequality satisfied by k. [5]

(ii) Given that $k = 2$, show that after B has collided with C there are no further collisions between any of the three spheres. [5]

40. [9231/s16/21/q1]

A bullet of mass 0.01 kg is fired horizontally into a fixed vertical barrier which exerts a constant resisting force of magnitude 1000 N. The bullet enters the barrier with speed $320\,\text{m s}^{-1}$ and emerges with speed $20\,\text{m s}^{-1}$. You may assume that the motion takes place in a horizontal straight line. Find

(i) the magnitude of the impulse that acts on the bullet, [2]

(ii) the thickness of the barrier, [2]

(iii) the time taken for the bullet to pass through the barrier. [1]

41. [9231/s16/21/q2]

A small smooth sphere A of mass m is moving with speed u on a smooth horizontal surface when it collides directly with an identical sphere B which is initially at rest on the surface. The coefficient of restitution between the spheres is e. Sphere B subsequently collides with a fixed vertical barrier which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the barrier is $\frac{1}{2}$. Given that 80% of the initial kinetic energy is lost as a result of the two collisions, find the value of e. [8]

42. [9231/s16/23/q5]

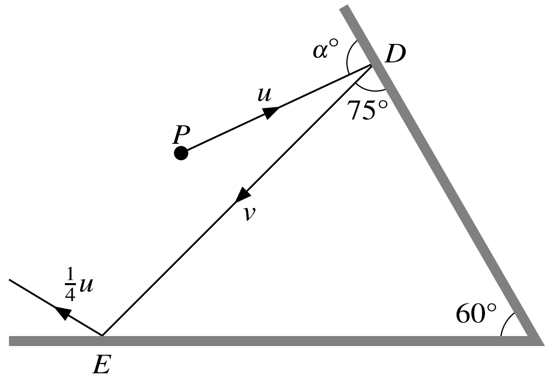
Three small smooth spheres A, B and C, of masses $5m$, m and km respectively, are at rest on a smooth horizontal surface. The spheres lie in a straight line with B between A and C. Sphere A is projected directly towards B with speed u. The coefficient of restitution between any pair of the spheres A, B and C is $\frac{4}{5}$. Show that the speed of A after the first collision is $\frac{7}{10}u$ and find the speed of B. [3]

Given that the speeds of A and C are equal after the second collision,

(i) show that the value of k is $\frac{20}{7}$, [4]

(ii) find the percentage loss in the kinetic energy of the system as a result of the two collisions. [5]

43. [9231/w16/21/q2]



Two smooth vertical walls each with their base on a smooth horizontal surface intersect at an angle of 60°. A small smooth sphere P is moving on the horizontal surface with speed u when it collides with the first vertical wall at the point D. The angle between the direction of motion of P and the wall is $\alpha°$ before the collision and 75° after the collision. The speed of P after this collision is v and the coefficient of restitution between P and the first wall is e. Sphere P then collides with the second vertical wall at the point E. The speed of P after this second collision is $\frac{1}{4}u$ (see diagram). The coefficient of restitution between P and the second wall is $\frac{3}{4}$.

(i) By considering the collision at E, show that $v = \dfrac{\sqrt{2}}{5}u$. [5]

(ii) Find the value of α and the value of e. [5]

44. [9231/w16/21/q4]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held vertically above O with the string taut and then projected horizontally with speed $\sqrt{\left(\frac{13}{3}ag\right)}$. It begins to move in a vertical circle with centre O. When P is at its lowest point, it collides with a stationary particle of mass λm. The two particles coalesce.

(i) Show that the speed of the combined particle immediately after the impact is $\dfrac{5}{\lambda + 1}\sqrt{\left(\frac{1}{3}ag\right)}$. [4]

In the subsequent motion, the string becomes slack when the combined particle is at a height of $\frac{1}{3}a$ above the level of O.

(ii) Find the value of λ. [6]

(iii) Find, in terms of m and g, the instantaneous change in the tension in the string as a result of the collision. [4]

45. [9231/s15/21/q5]

Three uniform small smooth spheres $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ have equal radii and masses $3\boldsymbol{m}$, $2\boldsymbol{m}$ and $\boldsymbol{m}$ respectively. The spheres are at rest in a straight line on a smooth horizontal surface, with $\boldsymbol{B}$ between $\boldsymbol{A}$ and $\boldsymbol{C}$. The coefficient of restitution between $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\boldsymbol{e}$ and the coefficient of restitution between $\boldsymbol{B}$ and $\boldsymbol{C}$ is $\boldsymbol{e}'$. Sphere $\boldsymbol{A}$ is projected directly towards $\boldsymbol{B}$ with speed $\boldsymbol{u}$. Show that, after the collision between $\boldsymbol{B}$ and $\boldsymbol{C}$, the speed of $\boldsymbol{C}$ is $\frac{2}{5}\boldsymbol{u}(1+\boldsymbol{e})(1+\boldsymbol{e}')$ and find the corresponding speed of $\boldsymbol{B}$. [7]

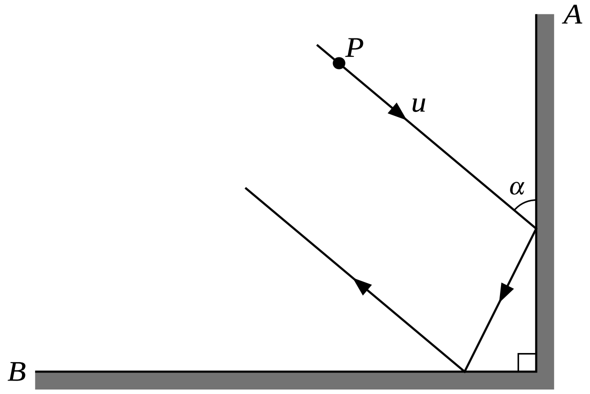
After this collision between $\boldsymbol{B}$ and $\boldsymbol{C}$ it is found that each of the three spheres has the same momentum. Find the values of $\boldsymbol{e}$ and $\boldsymbol{e}'$. [5]

46. [9231/s15/23/q1]

Two uniform small smooth spheres, A and B, of equal radii and masses 2 kg and 3 kg respectively, are at rest and not in contact on a smooth horizontal plane. Sphere A receives an impulse of magnitude 8 N s in the direction AB. The coefficient of restitution between the spheres is e. Find, in terms of e, the speeds of A and B after A collides with B. [5]

Given that the spheres move in opposite directions after the collision, show that $e > \frac{2}{3}$. [1]

47. [9231/s15/23/q2]



A uniform sphere ***P*** of mass ***m*** is at rest on a smooth horizontal table. The sphere is projected along the table with speed ***u*** and strikes a smooth vertical barrier ***A*** at an acute angle α. It then strikes another smooth vertical barrier ***B*** which is at right angles to ***A*** (see diagram). The coefficient of restitution between ***P*** and each of the barriers is ***e***. Show that the final direction of motion of ***P*** makes an angle $\frac{1}{2}\pi - \alpha$ with the barrier ***B*** and find the total loss in kinetic energy as a result of the two impacts. [7]

48. [9231/w15/21/q2]

A small uniform sphere A, of mass $2m$, is moving with speed u on a smooth horizontal surface when it collides directly with a small uniform sphere B, of mass m, which is at rest. The spheres have equal radii and the coefficient of restitution between them is e. Find expressions for the speeds of A and B immediately after the collision. [4]

Subsequently B collides with a vertical wall which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the wall is 0.4. After B has collided with the wall, the speeds of A and B are equal. Find e. [2]

Initially B is at a distance d from the wall. Find the distance of B from the wall when it next collides with A. [4]

49. [9231/w15/21/q4]

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When P is hanging at rest vertically below O, it is projected horizontally. In the subsequent motion P completes a vertical circle. The speed of P when it is at its highest point is u. Show that the least possible value of u is $\sqrt{(ag)}$. [2]

It is now given that $u = \sqrt{(ag)}$. When P passes through the lowest point of its path, it collides with, and coalesces with, a stationary particle of mass $\frac{1}{4}m$. Find the speed of the combined particle immediately after the collision. [4]

In the subsequent motion, when OP makes an angle θ with the upward vertical the tension in the string is T. Find an expression for T in terms of m, g and θ. [5]

Find the value of $\cos\theta$ when the string becomes slack. [2]

Formula Sheet MF19

Cambridge Assessment
International Education

List MF19

List of formulae and statistical tables

Cambridge International AS & A Level
Mathematics (9709) and Further Mathematics (9231)

For use from 2020 in all papers for the above syllabuses.

CST319

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PURE MATHEMATICS

Mensuration

Volume of sphere = $\frac{4}{3}\pi r^3$

Surface area of sphere = $4\pi r^2$

Volume of cone or pyramid = $\frac{1}{3} \times$ base area $\times$ height

Area of curved surface of cone = $\pi r \times$ slant height

Arc length of circle $= r\theta$ (θ in radians)

Area of sector of circle $= \frac{1}{2}r^2\theta$ (θ in radians)

Algebra

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d \,, \qquad S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}, \qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1), \qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n \text{, where } n \text{ is a positive integer}$$

$$\text{and } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{, where } n \text{ is rational and } |x| < 1$$

Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1+\tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1} x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Differentiation

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
uv	$v\frac{du}{dx} + u\frac{dv}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$	
x^n	$\frac{x^{n+1}}{n+1}$	$(n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x \rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	$(x > a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$	$(\lvert x \rvert < a)$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\lvert f(x)\rvert$$

Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

FURTHER PURE MATHEMATICS

Algebra

Summations:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)\,, \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\,, \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Trigonometry

If $t = \tan\tfrac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x \equiv 1\,, \qquad \sinh 2x \equiv 2\sinh x \cosh x\,, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathrm{d}x$	
$\sec x$	$\ln\lvert \sec x + \tan x \rvert = \ln\lvert \tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi)\rvert$	$\left(\lvert x \rvert < \tfrac{1}{2}\pi\right)$
$\operatorname{cosec} x$	$-\ln\lvert \operatorname{cosec} x + \cot x \rvert = \ln\lvert \tan(\tfrac{1}{2}x)\rvert$	$(0 < x < \pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\left(\lvert x \rvert < a\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	$(x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	

MECHANICS

Uniformly accelerated motion

$$v = u + at\,, \qquad s = \tfrac{1}{2}(u+v)t\,, \qquad s = ut + \tfrac{1}{2}at^2\,, \qquad v^2 = u^2 + 2as$$

FURTHER MECHANICS

Motion of a projectile

Equation of trajectory is:

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

Elastic strings and springs

$$T = \frac{\lambda x}{l}\,, \qquad E = \frac{\lambda x^2}{2l}$$

Motion in a circle

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$$\omega^2 r \qquad \text{or} \qquad \frac{v^2}{r}$$

Centres of mass of uniform bodies

Triangular lamina: $\tfrac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\tfrac{3}{8}r$ from centre

Hemispherical shell of radius r: $\tfrac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\dfrac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\dfrac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\tfrac{3}{4}h$ from vertex

PROBABILITY & STATISTICS

Summary statistics

For ungrouped data:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

For grouped data:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}, \qquad \text{standard deviation } = \sqrt{\frac{\Sigma(x-\bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

Discrete random variables

$$\mathrm{E}(X) = \Sigma xp, \qquad \mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2$$

For the binomial distribution $\mathrm{B}(n, p)$:

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \qquad \mu = np, \qquad \sigma^2 = np(1-p)$$

For the geometric distribution Geo(p):

$$p_r = p(1-p)^{r-1}, \qquad \mu = \frac{1}{p}$$

For the Poisson distribution $\mathrm{Po}(\lambda)$

$$p_r = \mathrm{e}^{-\lambda} \frac{\lambda^r}{r!}, \qquad \mu = \lambda, \qquad \sigma^2 = \lambda$$

Continuous random variables

$$\mathrm{E}(X) = \int x\, \mathrm{f}(x)\, \mathrm{d}x, \qquad \mathrm{Var}(X) = \int x^2\, \mathrm{f}(x)\, \mathrm{d}x - \{\mathrm{E}(X)\}^2$$

Sampling and testing

Unbiased estimators:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1} = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Central Limit Theorem:

$$\bar{X} \sim \mathrm{N}\left(\mu, \ \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$\mathrm{N}\left(p, \ \frac{p(1-p)}{n}\right)$$

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FURTHER PROBABILITY & STATISTICS

Sampling and testing

Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \overline{x}_1)^2 + \Sigma(x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$$

Probability generating functions

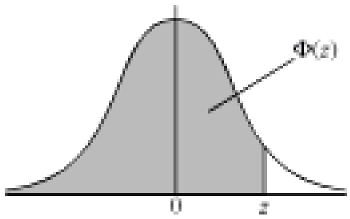
$\mathrm{G}_X(t) = \mathrm{E}(t^X)$, $\mathrm{E}(X) = \mathrm{G}'_X(1)$, $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \{\mathrm{G}'_X(1)\}^2$

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1, then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = P(Z \leqslant z).$$

For negative values of z, use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	ADD								
											1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1, then, for each value of p, the table gives the value of z such that

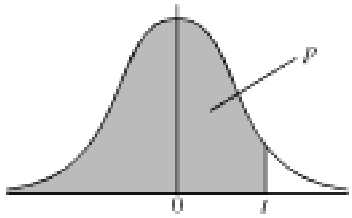
$$P(Z \leqslant z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE *t*-DISTRIBUTION

If T has a t-distribution with ν degrees of freedom, then, for each pair of values of p and ν, the table gives the value of t such that:

$P(T \leqslant t) = p$.

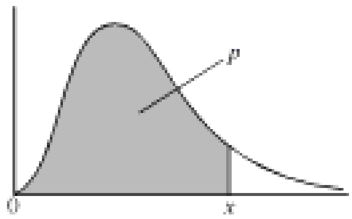


p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
$\nu = 1$	1.000	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.894	6.869
6	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.689
28	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.660
30	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE χ^2-DISTRIBUTION

If X has a χ^2-distribution with ν degrees of freedom then, for each pair of values of p and ν, the table gives the value of x such that

$P(X \leqslant x) = p$.



p	0.01	0.025	0.05	0.9	0.95	0.975	0.99	0.995	0.999
$\nu = 1$	0.0^31571	0.0^39821	0.0^23932	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

WILCOXON SIGNED-RANK TEST

The sample has size n.

P is the sum of the ranks corresponding to the positive differences.

Q is the sum of the ranks corresponding to the negative differences.

T is the smaller of P and Q.

For each value of n the table gives the **largest** value of T which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *T*

	Level of significance			
One-tailed	0.05	0.025	0.01	0.005
Two-tailed	0.1	0.05	0.02	0.01
$n = 6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

For larger values of n, each of P and Q can be approximated by the normal distribution with mean $\frac{1}{4}n(n+1)$ and variance $\frac{1}{24}n(n+1)(2n+1)$.

WILCOXON RANK-SUM TEST

The two samples have sizes m and n, where $m \leqslant n$.

R_m is the sum of the ranks of the items in the sample of size m.

W is the smaller of R_m and $m(n + m + 1) - R_m$.

For each pair of values of m and n, the table gives the **largest** value of W which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *W*

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 3$			$m = 4$			$m = 5$			$m = 6$		
3	6	–	–									
4	6	–	–	11	10	–						
5	7	6	–	12	11	10	19	17	16			
6	8	7	–	13	12	11	20	18	17	28	26	24
7	8	7	6	14	13	11	21	20	18	29	27	25
8	9	8	6	15	14	12	23	21	19	31	29	27
9	10	8	7	16	14	13	24	22	20	33	31	28
10	10	9	7	17	15	13	26	23	21	35	32	29

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m = 7$			$m = 8$			$m = 9$			$m = 10$		
7	39	36	34									
8	41	38	35	51	49	45						
9	43	40	37	54	51	47	66	62	59			
10	45	42	39	56	53	49	69	65	61	82	78	74

For larger values of m and n, the normal distribution with mean $\frac{1}{2}m(m+n+1)$ and variance $\frac{1}{12}mn(m+n+1)$ should be used as an approximation to the distribution of R_m.

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Syllabus 26-27 Further Mechanics

3 Further Mechanics (for Paper 3)

Knowledge of Cambridge International AS & A Level Mathematics (9709) Paper 4: Mechanics subject content is assumed for this component.

3.1 Motion of a projectile

Candidates should be able to:	Notes and examples
• model the motion of a projectile as a particle moving with constant acceleration and understand any limitations of the model	Vector methods are not required
• use horizontal and vertical equations of motion to solve problems on the motion of projectiles, including finding the magnitude and direction of the velocity at a given time or position, the range on a horizontal plane and the greatest height reached	
• derive and use the Cartesian equation of the trajectory of a projectile, including problems in which the initial speed and/or angle of projection may be unknown.	Knowledge of the 'bounding parabola' for accessible points is not included.

3.2 Equilibrium of a rigid body

Candidates should be able to:	Notes and examples
• calculate the moment of a force about a point	For questions involving coplanar forces only; understanding of the vector nature of moments is not required.
• use the result that the effect of gravity on a rigid body is equivalent to a single force acting at the centre of mass of the body, and identify the position of the centre of mass of a uniform body using considerations of symmetry	
• use given information about the position of the centre of mass of a triangular lamina and other simple shapes	Proofs of results given in the MF19 List of formulae are not required.
• determine the position of the centre of mass of a composite body by considering an equivalent system of particles	Simple cases only, e.g. a uniform L-shaped lamina, or a uniform cone joined at its base to a uniform hemisphere of the same radius.
• use the principle that if a rigid body is in equilibrium under the action of coplanar forces then the vector sum of the forces is zero and the sum of the moments of the forces about any point is zero, and the converse of this	
• solve problems involving the equilibrium of a single rigid body under the action of coplanar forces, including those involving toppling or sliding.	

3 Further Mechanics

3.3 Circular motion

Candidates should be able to:	Notes and examples
• understand the concept of angular speed for a particle moving in a circle, and use the relation $v = r\omega$	
• understand that the acceleration of a particle moving in a circle with constant speed is directed towards the centre of the circle, and use the formulae $r\omega^2$ and $\frac{v^2}{r}$.	Proof of the acceleration formulae is not required.
• solve problems which can be modelled by the motion of a particle moving in a horizontal circle with constant speed	
• solve problems which can be modelled by the motion of a particle in a vertical circle without loss of energy.	Including finding a normal contact force or the tension in a string, locating points at which these are zero, and condidtions for complete circular motion.

3.4 Hooke's law

Candidates should be able to:	Notes and examples
• use Hooke's law as a model relating the force in an elastic string or spring to the extension or compression, and understand the term modulus of elasticity	
• use the formula for the elastic potential energy stored in a string or spring	Proof of the formula is not required.
• solve problems involving forces due to elastic strings or springs, including those where considerations of work and energy are needed.	e.g. a particle moving horizontally or vertically or on an inclined plane while attached to one or more strings or springs, or a particle attached to an elastic string acting as a 'conical pendulum'.

3.5 Linear motion under a variable force

Candidates should be able to:	Notes and examples
• solve problems which can be modelled as the linear motion of a particle under the action of a variable force, by setting up and solving an appropriate differential equation.	Including use of $v\frac{\mathrm{d}v}{\mathrm{d}x}$ for acceleration, where appropriate. Calculus required is restricted to content from Pure Mathematics 3 in Cambridge International A Level Mathematics (9709). Only differential equations in which the variables are separable are included.

3 Further Mechanics

3.6 Momentum

Candidates should be able to:	Notes and examples
• recall Newton's experimental law and the definition of the coefficient of restitution, the property $0 \leqslant e \leqslant 1$, and the meaning of the terms 'perfectly elastic' ($e = 1$) and 'inelastic' ($e = 0$) • use conservation of linear momentum and/or Newton's experimental law to solve problems that may be modelled as the direct or oblique impact of two smooth spheres, or the direct or oblique impact of a smooth sphere with a fixed surface.	